ROTATION NUMBERS, PERIODIC POINTS AND TOPOLOGICAL ENTROPY OF A CLASS OF ENDOMORPHISMS OF THE CIRCLE

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SUMMARY.

In this thesis we consider the dynamics of a class of endomorphisms of the circle, we denote this class of functions by $\mathcal{A}$. Following the work of Milnor and Thurston [6] we develop a kneading theory for these functions, which enables us to calculate the entropy of our maps and to show that entropy is continuous.

We then show that any map, $f \in \mathcal{A}$, with positive entropy is topologically semi-conjugate to a piecewise linear map, F. The map F is determined by two real numbers, the topological entropy of f and the twist number of f, both of which can be calculated from the kneading matrix. Using the fact that the rotation intervals of f and F are the same, we give a method of calculating this interval from the twist number and entropy of f.

The final chapter is motivated by a theorem of of Sarkovskii [9] and concerns universal properties of the periodic points.

CONTENTS

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NOTATION

We denote each part of this thesis by three numbers (n,m,p) , where n gives the chapter number, m the section of that chapter and p the subsection. Whenever the thesis refers to another part of itself, these numbers are given in full. References to other sources are given in square brackets thus: [r] , where r refers to the position in the bibliography.

We introduce a substantial amount of notation that is non-standard, for convenience, the following list shows where a symbol is first introduced.

Symbol	Remarks	Reference
$\mathcal{A}$	class of endomorphisms of S^1	Introduction
$\rho(f,x)$	rotation number of x with respect to f	Introduction
$R(f)$	rotation interval of f	Introduction
$A(x)$	address of x	1.1.1
V	3-dimensional vector space with basis $\{I,II,III\}$	1.1.3
$\theta_*(x)$	kneading sequence associated to x	1.1.7
$\theta(x)$	invariant coordinate of x	1.1.11
$Q[[t]]$	ring of formal power series with rational coefficients	1.1.12
$V[[t]]$	module consisting of formal power series with coefficients in V	1.1.12
$\theta_I,\theta_{II},\theta_{III}$	invariant coordinate coefficients	1.1.12
$\nu(a),\nu(b),\nu(c)$	kneading invariants	1.2.3

D	kneading determinant	1.2.10, 2.2.8
$P = \bigcup_{n\geq 0} P_n$	pre-image set	1.3.1
$\gamma_c(J),\gamma_b(J)$		1.3.5
$\mathcal{A}^0_{b,c}$	subset of $\mathcal{A}$ with C^0-topology	2.0
$\mathcal{A}^1_{b,c}$	subset of $\mathcal{A}$ with C^1-topology	2.0
ent	entropy map	2.0
ν_c,ν_b	kneading invariant maps	2.2.3
$<<$	ordering on $Q[[t]]$	2.2.10
$S_{n,p}$	set of (n,p)-elements	3.2.1
$[X]$	orbit of $[X]$	3.2.3
$[X]\downarrow$	smallest element of $[X]$	3.2.4
$[X]\uparrow$	largest element of $[X]$	3.2.5
$gs(n,p)$	greatest smallest (n,p) element	3.2.6
$\ell\ell(n,p)$	least largest (n,p)-element	3.2.7
$n;p$	explicit (n,p)-element	3.2.16
$\psi(n;p)(\alpha,\beta)$	string of symbols α and β	3.2.20
$\theta_{n,p}$	specific element of $V[[t]]$	3.3.1
$\uparrow\theta,\downarrow\theta$	smallest and largest elements of an orbit in $V[[t]]$	3.3.3
$B_{n,p}$		3.4.1

I^{θ}, II^{θ}	elements of $V[[t]]$	3.4.3
$\mathcal{A}^{+}$	functions belonging to $\mathcal{A}$ with positive entropy	4.1.0
$\Lambda(J)$	measure on S^1	4.1.2
λ	map from S^1 to $\frac{[0,1]}{0\sim 1}$	4.1.7
$T(f)$	twist number	4.1.14
s	slope of piecewise linear map	4.3.6
$\theta_{II}(x,\frac{1}{s})$	real number associated to x	4.3.6
P_x	polynomial associated to x	4.3.10
$\psi_t(m,n)$	polynomial associated to $n;p$	4.3.14
$\lhd_n$	ordering on $n\times N$	5.0

INTRODUCTION.

In this thesis we shall consider the dynamics of a class of endomorphisms of the circle, $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. If we have a continuous map $f:S^1 \to S^1$ of degree one, then a lift of f is a continuous map $F:\mathbb{R} \to \mathbb{R}$ which covers f via the covering projection $\exp: t \to e^{2\pi it}$. Thus $f(e^{2\pi it}) = e^{2\pi iF(t)}$. It is easy to see that F satisfies $F(x+1) = F(x)+1$ and that two lifts of the same function f will differ by an integer translation.

We will now define the set, $\mathcal{A}$, of functions that we shall study.

DEFINITION. A continuous map $f:S^1 \to S^1$ of degree one belongs to $\mathcal{A}$ if there exists a lift, F, which has the following properties:

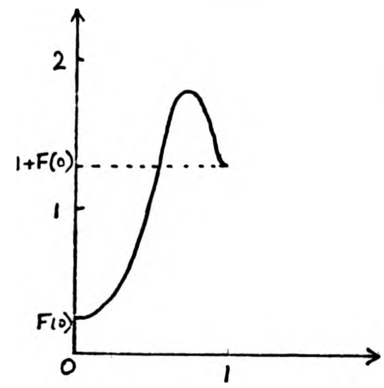
1) F has a minimum at 0.

2) F has one and only one maximum in $(0,1)$; occurring at b_F, say.

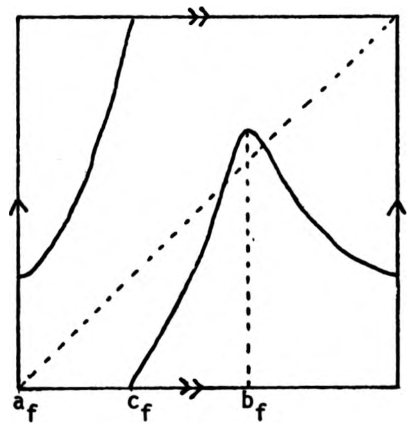
3) $F|_{[0,b_F]}$ is strictly monotone increasing and $F|_{[b_F,1]}$ is strictly monotone decreasing.

4) $F^{-1}(0)$ is a unique point.

Let f belong to $\mathcal{A}$ and let F be the lift with $0 \leq F(0) < 1$. Then the graph of $F|_{[0,1]}$ will look like the following.



Thus if f belongs to $\mathcal{A}$ it has one maximum and one minimum. Let $b_f \in S^1$ denote the point where the maximum occurs and $a_f \in S^1$ the point where the minimum occurs. It follows from the definition of A that $f^{-1}(a_f)$ is a unique point, which we denote by c_f ; if it is clear which function we are considering we will often not include the subscripts. We will draw the graph of f as follows.



The two main results of this thesis are contained in Chapters 4 and 5 and concern an explicit calculation of the rotation interval, $R(f)$, defined below, and some universal properties of the set of periodic orbits.

In the first chapter we develop a kneading theory as a tool for the study of these maps. In particular, we define a matrix, called the kneading matrix, associated to any function belonging to $\mathcal{A}$. In the second chapter we show how to calculate the topological entropy of $f \in \mathcal{A}$ from the determinant of the kneading matrix and go on to prove that topological entropy depends continuously on f (c.f. Misiurewicz and Szlenk [7]).

We will now briefly summarize some of the results from the classical theory of circle homeomorphisms which we extend to our class of functions.

If we are given a homeomorphism $f:S^1 \to S^1$, the lift F of f satisfying $0 \leq F(0) < 1$ and a point $x \in \mathbb{R}$ we can define the rotation number of x with respect to f to be $\lim_{n\to\infty} \frac{1}{n}(F^n(x)-x)$. This number is, in fact, independent of x and so we can associate a rotation number, $\rho(f)$, to f. If $\rho(f)$ is irrational then it can be shown that f is topologically semi-conjugate to a 'linear map' which is just rotation through $\rho(f)$. If $\rho(f)$ is rational then we can find a periodic point in S^1.

The analogous results for our endomorphisms are as follows. Given $f \in \mathcal{A}$, F the lift of f satisfying $0 \leq F(0) < 1$ and a point $x \in \mathbb{R}$, we define the rotation number of x to be $\limsup_{n\to\infty} \frac{1}{n}(F^n(x)-x)$. We will let $\rho(f,x)$ denote this number. Newhouse, Palis and Takens [8] have shown that $\rho(f,x)$ is not independent of x, so we cannot talk of the rotation number of f. However, they show that if we take $R(f) = \text{closure}\,\{\rho(f,x):x \in S^1\}$ we obtain a closed interval.* Moreover, if a rational number, $\frac{p}{q}$, belongs to $R(f)$ we can find $y \in S^1$ such that y is periodic with period q and $\rho(f,y) = \frac{p}{q}$. In the third chapter we prove that we can choose this point y so that it has a prescribed invariant coordinate.

* Ito [10] has since shown that $\{\rho(f,x):x \in S^1\}$ is closed.

In the fourth chapter we show that if f has positive entropy then f is topologically semi-conjugate to a piecewise linear map F. This map is determined by two numbers one of which is the entropy and the other is defined to be the twist number of f, denoted $T(f)$; like the entropy the twist number can be computed from the kneading determinant. We then prove that $R(f) = R(F)$ and so we can see that the entropy of f and $T(f)$ must determine the rotation interval. We conclude this chapter by using the results of Chapter 3 to give a method of calculating $R(f)$.

The fifth chapter is motivated by the following. Let $f:[0,1] \to [0,1]$ be any continuous map. Then there is an ordering of the positive integers, which is independent of f, such that if n_1 precedes n_2 and f has a periodic point of period n_2 then it has a periodic point of period n_1. Sarkovskii [9] proved this in 1964 (see 5.0 for a precise statement of the theorem). In Theorem 5.0.2 we prove an analogous result for our maps of the circle.

CHAPTER 1 KNEADING THEORY

1.0 INTRODUCTION

In this chapter we develop a kneading theory for our maps of the circle. This amounts to giving a symbolic representation of the dynamics of a map which is related to the way in which the map kneads (folds) the circle under iteration. Such a theory was developed by Milnor and Thurston in [6] for continuous piecewise monotone maps of the interval or real line. This chapter and the next follow this work closely adapting it to suit our problem.

We will order points in $S^1 \setminus \{a\}$ in an anti-clockwise way. More precisely, let x and y belong to $S^1 \setminus \{a\}$, suppose $x = e^{2\pi it}$ and $y = c^{2\pi it'}$ then we will say $x < y$ if $t(\text{mod } 1) < t'(\text{mod } 1)$. Thus $b > c$ and we shall let (c,b) denote the interval $\{x \in S^1 : c < x < b\}$.

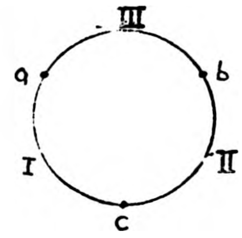
In the first section we show how, when given $f \in \mathcal{A}$, to associate to each point, $x \in S^1$, a formal power series called the invariant coordinate of x and denoted $\theta(x)$. We then show that the map θ is order preserving.

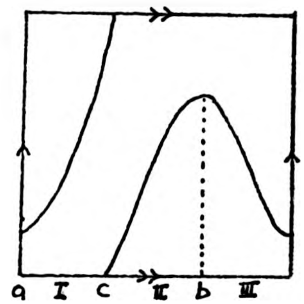
Then in the second section we look at $\theta(x)$ for points near to the critical points of f and define a matrix called the kneading matrix of f . This matrix contains a large amount of information about f ; in later chapters we show that we can calculate the entropy and compute the rotation interval of f when given its kneading matrix.

The third section shows how the cardinality of the pre-image points b and c is related to the kneading matrix. This is important for the calculation of entropy which is done in the second chapter.

1.1 INVARIANT COORDINATES

1.1.0 We will denote the intervals (a,c) by I , (c,b) by II and (b,a) by III.





GRAPH OF f .

1.1.1 DEFINITION. The <u>address</u>, $A(x)$, of a point x in S^1 is defined by the following. $A(x) = I$ if $x \in (a,c)$, II if $x \in (c,b)$, III if $x \in (b,a)$, $\frac{I+III}{2}$ if $x = a$, $\frac{III+II}{2}$ if $x = b$ and $\frac{I+II}{2}$ if $x = c$.

1.1.2 DEFINITION. The <u>itinerary</u> of a point x is the infinite string of symbols given by $(A(x)\ , A(f(x))\ , A(f^2(x)),\ldots)$.

1.1.3 DEFINITION. Let V be the three dimensional vector space over the rationals with basis $\{I,II,III\}$.

1.1.4 DEFINITION. Let $A:S^1 \to V$ denote the map which sends a point x to its address.

1.1.5 Let $\varepsilon:V \to \mathbb{R}$ be the linear map such that $\varepsilon(I) = 1$, $\varepsilon(II) = 1$ and $\varepsilon(III) = -1$.

1.1.6 We can now construct a map from S^1 to $\mathbb{R}$ by $S^1 \xrightarrow{A} V \xrightarrow{\varepsilon} \mathbb{R}$. Simple calculation, using the fact that ε is linear, shows that $\varepsilon(A(a)) = 0$, $\varepsilon(A(b)) = 0$ and $\varepsilon(A(c)) = 1$. We have defined ε so that $\varepsilon(A(x))$ is positive if f is orientation preserving at x , negative if f is orientation reversing at x and zero if x is a maximum or minimum.

1.1.7 DEFINITION. Let x belong to S^1 and let its itinerary be $(A_0,A_1,\ldots)$ .. Then we define the <u>kneading sequence</u>, $\theta_*(x)$, associated to x to be the infinite string $(\theta_0,\theta_1,\theta_2,\ldots)$ where $\theta_0 = A_0$, $\theta_1 = \varepsilon(A_0) \times A_1$, $\theta_2 = \varepsilon(A_1) \times \varepsilon(A_0) \times A_2,\ldots,$ $\theta_n = \varepsilon(A_{n-1}) \times \varepsilon(A_{n-2})\ldots\times\varepsilon(A_0) \times A_n$.

1.1.8 EXAMPLE. Suppose x has the following properties.

1.2 THE KNEADING MATRIX

1.2.0 Given a function $g:\mathbb{R} \to \mathbb{R}$ we can talk about right and left-hand limits. Analogously, given $f:S^1 \to S^1$ we will consider clockwise and anti-clockwise limits.

1.2.1 DEFINITION. Let $\theta(x+)$ denote the limit of $\theta(y)$ as y tends to x in an anti-clockwise way and let $\theta(x-)$ denote the limit of $\theta(y)$ as y tends to x in a clockwise way.

1.2.2 LEMMA. $\theta(x-)$ _and_ $\theta(x+)$ _are well-defined_.

Proof. We can choose $\varepsilon_0 > 0$ such that $(x-\varepsilon_0,x) \cap \{a,b,c\} = \emptyset$. Similarly, we can choose $\varepsilon_1 > 0$ with $\varepsilon_1 < \varepsilon_0$ such that $\{f^{-1}(a) \cup f^{-1}(b) \cup f^{-1}(c)\} \cap (x-\varepsilon_1,x) = \emptyset$. Inductively, we can find an $\varepsilon_n > 0$ such that

$$\{\bigcup_0^n f^{-i}(a) \cup \bigcup_0^n f^{-i}(c) \cup \bigcup_0^n f^{-i}(b)\} \cap (x-\varepsilon_n,x) = \emptyset .$$

This implies that if z and y belong to $(x-\varepsilon_n,x)$ then $\theta(z)$ is congruent to $\theta(y)$ modulo t^{n+1} . So the limit $\theta(x-)$ exists.

A similar proof works for $\theta(x+)$.

1.2.3 DEFINITION. The _kneading invariants_ $\nu(a)$, $\nu(b)$ and $\nu(c)$ are defined by $\nu(a) = \theta(a+) - \theta(a-)$, $\nu(b) = \theta(b+) - \theta(b-)$ and $\nu(c) = \theta(c+) - \theta(c-)$.

1.2.4 LEMMA. $t\nu(a) = \nu(c) - II + I$.

Proof. $\theta(c+) = II + t\theta(f(c)+) = II + t\,\theta(a+)$.

Similarly $\theta(c-) = I + t\,\theta(a-)$. Subtracting gives $\theta(c+) - \theta(c-) = t(\theta(a+) - \theta(a-)) + II - I$ which completes the proof.

1.2.5 LEMMA. $\theta(a+) = \theta(a) + \frac{\nu(a)}{2}$; and

$$\theta(a-) = \theta(a) - \frac{\nu(a)}{2} .$$

Proof. The invariant coordinates of $a+$ and $a-$ are well-defined. So we can consider the kneading sequences of $a+$ and $a-$. Let $\theta_*(a+) = (I, A_1, k_2A_2, k_3A_3, \ldots, k_nA_n, \ldots)$ where k_i is either $+1$ or -1 . Then it is easily seen that $\theta_*(a-) = (III, -A_1, -k_2A_2, \ldots, -k_nA_n, \ldots)$. Therefore $\nu(a) = I - III + 2A_1t + 2k_2A_2t + \ldots + 2k_nA_nt \ldots$. Using the fact that $2\theta(a) = I + III$ it is easy to verify that $\nu(a)$ equals $2\theta(a+) - 2\theta(a)$ and $2\theta(a) - 2\theta(a-)$.

1.2.6 LEMMA. $\theta(b+) = \theta(b) + \frac{\nu(b)}{2}$

$$\theta(b-) = \theta(b) - \frac{\nu(b)}{2} .$$

Proof. The proof is similar to the previous proof.

1.2.7 LEMMA. $\theta(c+) = \theta(c) + \frac{\nu(c)}{2}$

$$\theta(c-) = \theta(c) - \frac{\nu(c)}{2} .$$

Proof. $\theta(c+) = II + t\theta(a+)$ and $\theta(c) = \frac{I+II}{2} + t\theta(a)$.

Subtracting the second equation from the first and using Lemma 1.2.5 gives $\theta(c+) - \theta(c) = \frac{II-I}{2} + t\,\frac{\nu(a)}{2}$. Lemma 1.2.4 tells us that $\frac{II-I}{2} + t\,\frac{\nu(a)}{2} = \frac{\nu(c)}{2}$.

Similarly we can show that $\theta(c) - \theta(c-) = \frac{\nu(c)}{2}$.

1.2.8 Since $\nu(c)$ belongs to $V[[t]]$ we can express it in the form $\nu^I(c)I + \nu^{II}(c)II + \nu^{III}(c)III$, where the coefficients are formal power series with rational coefficients.

Similarly, we have $\nu(b) = \nu^I(b)I + \nu^{II}(b)II + \nu^{III}(b)III$ and $\nu(c) = \nu^I(c)I + \nu^{II}(c)II + \nu^{III}(c)III$.

1.2.9 DEFINITION. The kneading matrix is defined to be

$$\begin{vmatrix} \nu^{I}(c) & \nu^{I}(b) \\ \nu^{II}(c) & \nu^{II}(b) \end{vmatrix}$$

1.2.10 DEFINITION. The kneading determinant, D, is the determinant of the kneading matrix. i.e. $D = \nu^{I}(c)\nu^{II}(b)-\nu^{I}(b)\nu^{II}(c)$.

1.2.11 LEMMA. $(1-t)\nu^{I}(b)+(1-t)\nu^{II}(b)+(1+t)\nu^{III}(b) = 0$

$$(1-t)\nu^{I}(c)+(1-t)\nu^{II}(c)+(1+t)\nu^{III}(c) = 0$$

$$(1-t)\nu^{I}(a)+(1-t)\nu^{II}(a)+(1+t)\nu^{III}(a) = 0 \ .$$

Proof. The proof follows easily from lemma 1.1.13.

1.3 THE PRE-IMAGE SET

1.3.1 DEFINITION. For all $n \geq 0$ let P_n denote the set $\{x \in S^1 | f^n(x) \in \{a,b,c\}$ and $f^j(x) \notin \{a,b,c\}$ for $0 \leq j < n\}$ and let P denote $\bigcup_{n\geq 0} P_n$. This last set is called the pre-image set.

1.3.2 We make two observations. Firstly, if x belongs to P_i , where i is strictly greater than zero, then $f^i(x)$ cannot equal a , because this would imply that $f^{i-1}(x) = c$ and so x would belong to P_{i-1} . Secondly, P_0 is $\{a,b,c\}$.

1.3.3 LEMMA. Let x belong to $S^1\backslash P$. Then θ is continuous at x .

Proof. Let $i \geq 0$ be an integer. Since x is not an element of the pre-image set we can find an open neighbourhood U_i of x with the property that if y belongs to U_i then $f^i(y)$ is not equal to a,b or c .

Let n be a positive integer. Then it is clear that $\bigcap_{i=0}^{n} U_i$ is an open neighbourhood of x , and that if y belongs to $\bigcap_{i=0}^{n} U_i$ then $\theta(x) \equiv \theta(y)$ modulo n . Therefore θ must be continuous at x .

1.3.4 LEMMA. Let x belong to P_n . Then we have

$$\theta(x+) = \theta(x) + \frac{t^n \nu(f^n(x))}{2} \quad \text{and} \quad \theta(x-) = \theta(x) - \frac{t^n \nu(f^n(x))}{2} \quad .$$

Proof. Lemma 1.1.10 tells us that $\theta(x+) \geq \theta(x) \geq \theta(x-)$. So the proof follows easily from lemmas 1.2.5, 1.2.6 and 1.2.7.

1.3.5 DEFINITION. Let $J \subsetneq S^1$ be an interval. We will define two elements of $\mathbb{Z}[[t]]$, denoted by $\gamma_c(J)$ and $\gamma_b(J)$ by the following. First, $\gamma_c(J) = \sum_{i=0}^{\infty} \gamma_i t^i$ where γ_i is the number of elements in $\{x \in J \cap P_i \mid f^i(x) = c\}$. Secondly, $\gamma_b(J) = \sum_{i=0}^{\infty} \gamma_i^1 t^i$ where γ_i^1 is the number of elements in $\{x \in J \cap P_i \mid f^i(x) = b\}$.

1.3.6 LEMMA. Let $[e,f] \subset S^1 \setminus \{a\}$ be a closed interval. Let θ be continuous at both e and f . Then we can compute $\theta(f)-\theta(e)$ by summing the discontinuities $\theta(x+)-\theta(x-)$ over all points, x , belonging to (e,f) .

Proof. $\theta(x)$ modulo t^n is a step function with only finitely many discontinuities in $[e,f]$. Therefore $\theta(f)-\theta(e)$ is congruent to $\sum_{x \in (e,f)} \theta(x+)-\theta(x-)$ modulo t^n . Taking the limit as n tends to infinity gives the required result.

1.3.7 LEMMA. Let $[e,f] \subset S^1 \setminus \{a\}$ be a closed interval. Let θ be continuous at both e and f . Then we have

$$\theta(f) - \theta(e) = \gamma_c([e,f])\nu(c) + \gamma_b([e,f])\nu(b) \ .$$

Proof. Let x belong to $[e,f]$. Suppose that x also belongs to P_n .

Then we know that $f^n(x)$ equals either b or c (see 1.3.3). So x makes a contribution of t^n to $\gamma_{f^n(x)}[e,f]$ and hence a contribution of $t^n\nu(f^n(x))$ to $\gamma_{f^n(x)}([e,f])\nu(f^n(x))$. But lemma 1.3.4 tells us that $\theta(x+) - \theta(x-)$ equals $t^n\nu(f^n(x))$. Summing over all x belonging to $(e,f) \cap P$ completes the proof.

If we drop the hypothesis that θ is continuous at both e and f the following is easily seen to be true.

1.3.8 LEMMA. Let $[e,f] \subset S^1$ be a closed interval. Then we have $\theta(f-) - \theta(e+) = \gamma_c([e,f])\nu(c) + \gamma_b([e,f])\nu(b)$.

1.3.9 In particular it is easy to check that

$$\gamma_c(S^1)\nu(c) + \gamma_b(S^1)\nu(b) = \theta(a-) - \theta(a+) \quad .$$

Recall that $\{I,II,III\}$ is a basis for $V[[t]]$. So comparing coefficients we can split the above equation into the following three equations.

$$\gamma_c(S^1)\nu^I(c) + \gamma_b(S^1)\nu^I(b) = \theta_I(a-) - \theta_I(a+) \quad ,$$

$$\gamma_c(S^1)\nu^{II}(c) + \gamma_b(S^1)\nu^{II}(b) = \theta_{II}(a-) - \theta_{II}(a+)$$

$$\gamma_c(S^1)\nu^{III}(c) + \gamma_b(S^1)\nu^{III}(b) = \theta_{III}(a-) - \theta_{III}(a+) \quad .$$

In matrix form this gives

$$\begin{vmatrix} \nu^I(c) & \nu^I(b) \\ \nu^{II}(c) & \nu^{II}(b) \end{vmatrix} \begin{vmatrix} \gamma_c(S^1) \\ \gamma_b(S^1) \end{vmatrix} = \begin{vmatrix} \theta_I(a-)-\theta_I(a+) \\ \theta_{II}(a-)-\theta_{II}(a+) \end{vmatrix} \quad .$$

CHAPTER 2 TOPOLOGICAL ENTROPY

2.0 INTRODUCTION

In this chapter we will again follow the work of Milnor and Thurston. We study the various formal power series, that have been introduced in the first chapter, thinking of them as convergent power series. Thus t is no longer regarded as a formal symbol but as a complex variable.

For example, if we are given $f \in \mathcal{A}$ we have shown how to construct its kneading determinant $D(f)$. Let the kneading matrix associated to f be

$$\begin{vmatrix} \nu^{I}(c) & \nu^{I}(b) \\ \nu^{II}(c) & \nu^{II}(b) \end{vmatrix} ;$$

then it is easy to see that each entry of the matrix converges if $|t| < 1$. So we can think of the entries as being complex analytic functions defined on the unit disc. It is clear that $D(f)$ can also be considered as being complex analytic on the unit disc. We will show that this function has a positive real zero in $[0,1]$ and the least such zero , r, determines the the topological entropy h of f via the formula $h= -\log r$.*

Let $\mathcal{A}^{0}_{b,c}$ denote $\{f \in \mathcal{A}: b_f = b, c_f = c\}$ and let $\mathcal{A}^{1}_{b,c}$ be the set of C^1-functions belonging to $\mathcal{A}^{0}_{b,c}$. The C^0-topology on $\mathcal{A}^{0}_{b,c}$ is that induced by the norm $|f|_0 = \sup_{x \in S^1} ||f(x)||$; the norm $|f|_1 = \sup_{x \in S^1} (||f(x)|| + ||df(x)||)$ induces the C^1-topology on $\mathcal{A}^{1}_{b,c}$. Here $||\cdot||$ denotes the usual norm on $\mathbb{C}$ and the induced norm on its cotangent space and we regard S^1 as a C^∞-submanifold of $\mathbb{C}$.

In the first section we show how to calculate the topological entropy, $h(f)$,

* For the definition of topological entropy see Appendix p.83.

of $f \in \mathcal{A}$ from its kneading determinant. Then, in the second section, we show that the entropy map, $ent: \mathcal{A}^0_{b,c} \to R$, is continuous in the C^0-topology at any function for which neither c nor b are periodic points. In the third section we prove that $ent|_{\mathcal{A}^1_{b,c}}$ is continuous in the C^1-topology.

2.1 CALCULATION OF TOPOLOGICAL ENTROPY

2.1.0 The aim of this section is to show how to calculate the topological entropy of a map belonging to $\mathcal{A}$ from its kneading determinant. First, we will show how the entropy of a map is related to the radius of convergence of $\gamma_b(S^1)+\gamma_c(S^1)$. The fact that the topological entropy is related to the rate of growth of the numbers C_n, defined below, is proved by Misiurewicz and Szlenk in [7], and our result could be proved as a corollary to this. However, we give here a self-contained elementary proof following the ideas due to Lai-Sang Young contained in [2].

2.1.1 DEFINITION. A finite sequence $A_0A_1\ldots A_m$ of symbols I,II,III is called <u>admissible</u> if there exists an $x \in S^1$ whose itinerary begins $A_0A_1\ldots A_m$.

2.1.2 DEFINITION. Let $\Sigma(f)$ denote the set $\{\underline{x} \in \prod_0^\infty \{I,II,III\} : x_0x_1\ldots x_m$ is admissible for every $m\}$.

2.1.3 DEFINITION. Let C_n denote the cardinality of $\{A_0A_1\ldots A_{n-1} : A_0A_1\ldots A_{n-1}$ is admissible$\}$.

2.1.4 LEMMA. $\Sigma C_n t^n = t(1+\gamma_b(S^1)+\gamma_c(S^1))(1-t)^{-1}$.

<u>Proof</u>. Let $\bar{p}: [0,1) \to S^1$ denote the map which sends x to $e^{2\pi ix}$. Then $p^{-1} o f o p$ is a map from $[0,1)$ to itself. Let $A_0A_1\ldots A_{n-1}$ be admissible; then we can see that it corresponds to a maximal interval on which $p^{-1} o f^n o p$ is strictly monotone. So C_n is the number of distinct maximal open intervals for which $p^{-1} o f^n o p$ is strictly monotone.

We will now calculate C_n. Suppose that $x \in (0,1)$ is either a local maximum or local minimum of $p^{-1}of^n op$. Then either $f^m(x) = a$ for some m satisfying $0 < m \leq n$, or $f^p(x) = b$ for some p satisfying $0 \leq p \leq n$.

Let $\gamma_c(S^1) = \sum_0^\infty \gamma_c^i t^i$ and $\gamma_b(S^1) = \sum_0^\infty \gamma_b^i t^i$. Then it is easily seen that the number of local maxima plus the number of local minima of $p^{-1}of^n op$ in $(0,1)$ is $\sum_0^{n-1} \gamma_c^i + \sum_0^n \gamma_b^i$. The number of discontinuities of $p^{-1}of^n op$ is γ_c^n. So we have $C_n = 1+\sum_0^n \gamma_c^i + \sum_0^n \gamma_b^i$. Thus $\sum_1^\infty C_n t^n = t(1+\gamma_b(S^1)+\gamma_c(S^1))(1-t)^{-1}$.

2.1.5 DEFINITION. Let $\underline{x} = (A_0, A_1, \ldots)$ belong to $\Sigma(f)$. Then we will let $I(\underline{x})$ denote $\bigcap_{n=0}^\infty$ closure $(\bigcap_{j=0}^n f^{-j}A_j)$.

We will now state two result of Rufus Bowen. For proof of 2.1.6 see [4] and proof of 2.1.7 see [3].

2.1.6 LEMMA. <u>Suppose</u> $\{a_m\}_{m=1}^\infty$ <u>is a sequence satisfying</u> $\inf \frac{a_m}{m} > -\infty$ <u>and</u> $a_{n+m} \leq a_n + a_m$. <u>Then</u> $\lim_{m\to\infty} \frac{a_m}{m}$ exists.

2.1.7 PROPOSITION. <u>Let</u> X <u>and</u> Y <u>be compact metric spaces and</u> $T: X \to X$, $S: Y \to Y$, $\pi: X \to Y$ (surjective) be continuous maps with $\pi o T = S o \pi$. Suppose that for any $y \in Y$ <u>we have</u> $h(T, \pi^{-1}(y)) = 0$. <u>Then</u> $h(T) = h(S)$.

We need these results to prove the following.

2.1.8 PROPOSITION. <u>Let</u> f <u>belong to</u> $\mathcal{A}$. <u>Let</u> r <u>be the radius of convergence of</u> $\gamma_c(S^1) + \gamma_b(S^1)$. <u>Then</u> $\log(\frac{1}{r}) = h(f)$.

<u>Proof</u>. Let $\sigma: \Sigma(f) \to \Sigma(f)$ denote the shift map which sends $A_0A_1\ldots$ to $A_1A_2\ldots$. Then from the definition of entropy we obtain $h(\sigma) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log C_n$ and by Lemma 2.1.6 this must equal $\lim_{n\to\infty} \frac{1}{n} \log C_n$. The radius of convergence of $\Sigma C_n t^n$ is $r^1 = \lim_{n\to\infty} \frac{1}{(C_n)^{1/n}}$. It is easily seen from Lemma 2.1.4 that $r^1 = r$.

So $h(\sigma) = \log\left(\frac{1}{r}\right)$.

To complete the proof we need to show that $h(\sigma) = h(f)$. Let $\tilde{\Sigma} = \{(\underline{x},x) \in \Sigma(f) \times S : x \in I(\underline{x})\}$, let $\pi_1(\underline{x},x) = \underline{x}$ and $\pi_2(\underline{x},x) = x$. Then we define $\tilde{\sigma}:\tilde{\Sigma} \to \tilde{\Sigma}$ by $(\underline{x},x) \to (\sigma(\underline{x}),f(x))$. Thus $\tilde{\sigma}$ covers $\sigma|_\Sigma$ by π_1 and $f|_{S^1}$ by π_2 .

$$\begin{tikzcd}
S^1 \arrow[r, "f"] & S^1 \\
\tilde{\Sigma} \arrow[u, "\pi_2"] \arrow[r, "\tilde{\sigma}"] \arrow[d, "\pi_1"] & \tilde{\Sigma} \arrow[u, "\pi_2"] \arrow[d, "\pi_1"] \\
\Sigma \arrow[r, "\sigma"] & \Sigma
\end{tikzcd}$$

Since π_2 is at most two to one we have, by 2.1.6, that $h(f) = h(\tilde{\sigma})$. The map π_1 collapses $I(\underline{x})$ to a point, but $f^n|_{I(\underline{x})}$ is a homeomorphism for all n , so $h(f,I(\underline{x})) = 0$ and thus $h(\tilde{\sigma},\pi_1^{-1}(\underline{x})) = 0$ and by Proposition 2.1.7 we have $h(\tilde{\sigma}) = h(\sigma)$. So we have shown that $h(\sigma) = h(f)$ and completed the proof.

2.1.9 The rest of this section will be devoted to showing that the radius of convergence of $\gamma_c(S^1) + \gamma_b(S^1)$ is equal to the smallest positive zero of the kneading determinant.

2.1.10 LEMMA. <u>The series</u> $\gamma_b(S^1)$ <u>and</u> $\gamma_c(S^1)$ <u>extend to meromorphic functions throughout the unit disc. These meromorphic functions have poles at most at the zeroes of</u> D .

<u>Proof.</u> We know from 1.3.9 that we have the following equation.

$$\begin{vmatrix} \nu^{I}(c) & \nu^{I}(b) \\ \nu^{II}(c) & \nu^{II}(b) \end{vmatrix} \quad \begin{vmatrix} \gamma_c(S^1) \\ \gamma_b(S^1) \end{vmatrix} = \begin{vmatrix} \theta_I(a-) - \theta_I(a+) \\ \theta_{II}(a-) - \theta_{II}(a+) \end{vmatrix}$$

Since $\nu^{I}(c)$ and $\nu^{II}(b)$ both start with -1 and $\nu^{I}(b)$ starts with 0, the kneading matrix is invertible in $Z[[t]]$. Thus

$$\begin{vmatrix} \gamma_c(S^1) \\ \gamma_b(S^1) \end{vmatrix} = \begin{vmatrix} \nu^{I}(c) & \nu^{I}(b) \\ \nu^{II}(c) & \nu^{II}(b) \end{vmatrix}^{-1} \quad \begin{vmatrix} \theta_I(a-) - \theta_I(a+) \\ \theta_{II}(a-) - \theta_{II}(a+) \end{vmatrix} .$$

Multiplying throughout by the analytic function D gives

$$\begin{vmatrix} D\gamma_c(S^1) \\ D\gamma_b(S^1) \end{vmatrix} = D \begin{vmatrix} \nu^{I}(c) & \nu^{I}(b) \\ \nu^{II}(c) & \nu^{II}(b) \end{vmatrix}^{-1} \quad \begin{vmatrix} \theta_I(a-) - \theta_I(a+) \\ \theta_{II}(a-) - \theta_{II}(a+) \end{vmatrix} .$$

This shows that $D\gamma_c(S^1)$ and $D\gamma_b(S^1)$ are the products of functions which are analytic for $|t| < 1$.

2.1.11 COROLLARY. <u>The kneading determinant</u> D <u>has a zero at</u> r.

<u>Proof</u>. Since $\gamma_c(S^1) + \gamma_b(S^1)$ has a pole at r, either $\gamma_c(S^1)$ or $\gamma_b(S^1)$ must have a pole at r. But $D\gamma_c(S^1)$ and $D\gamma_b(S^1)$ are analytic throughout the unit disc, so D must have a zero at r.

2.1.12 LEMMA. <u>The kneading determinant is non-zero for</u> $|t| < r$.

<u>Proof</u>. Lemma 1.3.8 tells us that $\theta(c-) - \theta(a+)$ is equal to $\gamma_c([a,c])\nu(c)+\gamma_b([a,c])\nu(b)$. From lemmas 1.2.5 and 1.2.7 we can see that

$$\theta(c) - \theta(a) = \gamma_c([a,c])\nu(c) + \gamma_b([a,c])\nu(b) + \tfrac{1}{2}\nu(c) + \tfrac{1}{2}\nu(a) \quad .$$

Thus by lemma 1.2.4 we obtain

$$t[\theta(c)-\theta(a)] = t\gamma_c([a,c])\nu(c) + t\gamma_b([a,c])\nu(b) + \tfrac{1}{2}t\nu(c) + \tfrac{1}{2}(\nu(c)-II+I) \quad .$$

Using the facts that $\theta(c) = \frac{II+I}{2} + t\frac{I+III}{2}$ and $\theta(a) = \frac{I+III}{2}$ we can deduce the following equation.

$$\text{(A)}\ \tfrac{1}{2}(II-I)+\tfrac{1}{2}t\,[II-III+t(I+III)] = [t\gamma_c([a,c])+\tfrac{1}{2}+t]\nu(c)+t\gamma_b([a,c])\nu(b) \quad .$$

Similarly we can obtain

$$\frac{II-I}{2} + t[\theta(b)-\theta(a)] = [t\gamma_c([a,b])+\tfrac{1}{2}]\nu(c)+t[\gamma_b([a,c])+\tfrac{1}{2}]\nu(b) \ .$$

This reduces to

$$\text{(B)}\ \tfrac{1}{2}(II-I)+\tfrac{1}{2}t\,[II-I] = [t\gamma_c([a,b])+\tfrac{1}{2}]\nu(c)+t[\gamma_b([a,c])+\tfrac{1}{2}]\nu(b) \quad .$$

Equating coefficients in $V[[t]]$ and rewriting equations A and B in matrix notation gives

$$\begin{vmatrix} -\tfrac{1}{2}+t^2 & \tfrac{1}{2}+\tfrac{1}{2}t \\ -\tfrac{1}{2}-\tfrac{1}{2}t & \tfrac{1}{2}+\tfrac{1}{2}t \end{vmatrix} = \begin{vmatrix} t\gamma_c([a,c])+\tfrac{1}{2}+\tfrac{1}{2}t & -t\gamma_b([a,c]) \\ t\gamma_c([a,c])+\tfrac{1}{2} & t\gamma_b([a,b])+\tfrac{1}{2} \end{vmatrix} \begin{vmatrix} \nu^{I}(c) & \nu^{II}(c) \\ \nu^{I}(b) & \nu^{II}(b) \end{vmatrix} .$$

The determinant of $\begin{vmatrix} -\tfrac{1}{2}+t^2 & \tfrac{1}{2}+\tfrac{1}{2}t \\ -\tfrac{1}{2}-\frac{t}{2} & \tfrac{1}{2}+\tfrac{1}{2}t \end{vmatrix}$ is $\tfrac{1}{4}t(1+t)^2$

which is non-zero on the unit disc, except when $t = 0$.

The determinant of $\begin{vmatrix} t\gamma_c([a,c])+\tfrac{1}{2}+\tfrac{1}{2}t & t\gamma_b([a,c]) \\ t\gamma_c([a,b])+\tfrac{1}{2} & t\gamma_b([a,b])+\tfrac{1}{2}t \end{vmatrix}$ is defined and analytic for $|t| < r$. Since $D(0) = 1$, the kneading determinant must be non-zero for $|t| < r$.

2.1.13 We have now shown the following to be true.

THEOREM. Let f belong to $\mathcal{A}$. Let r be the smallest positive real zero of the kneading determinant. Then $\log(\frac{1}{r})$ is the topological entropy of f .

2.2 CONTINUITY OF ENTROPY IN $\mathcal{A}^o_{b,c}$.

2.2.0 In this section we prove that entropy is continuous at any function, f , belonging to $\mathcal{A}^o_{b,c}$ for which c and b are not periodic points.

2.2.1 In what follows we shall use the following standard result from analysis (for proof see Abraham and Robbin [1], p.25).

LEMMA. Let n be a positive integer. Let $\phi_n : C^r(S^1,S^1)\times S^1 \to C^r(S^1,S^1) \times S^1$ be the map defined by (f,x) goes to $(f,f^n(x))$. Then ϕ_n is C^r .

2.2.2 NOTATION. Let f and g belong to $\mathcal{A}^o_{b,c}$. Let x belong to S^1 . Then the invariant coordinate of x with respect to f will be denoted $\theta_f(x)$ and the invariant coordinate of x with respect to g will be denoted $\theta_g(x)$.

2.2.3 DEFINITION. Let $\nu_c : \mathcal{A}^o_{b,c} \to V[[t]]$ be the map which sends g to its kneading invariant $\nu(c)$. Similarly, let $\nu_b : \mathcal{A}^o_{b,c} \to V[[t]]$ be the map which sends g to its kneading invariant $\nu(b)$.

2.2.4 In the next three lemmas we indicate how $\nu(b)$ and $\nu(c)$ change as f is perturbed in $\mathcal{A}^o_{b,c}$.

2.2.5 LEMMA. Let $f \in \mathcal{A}^o_{b,c}$ and suppose that f satisfies the following properties:

(i) c and b are not periodic points; and

(ii) for all $n \geq 1$, $f^n(b) \neq c$ and $f^n(c) \neq b$.

Then ν_c and ν_b are continuous at f.

Proof. Let n be any positive integer. We can see from lemma 2.2.1 that there exists a small neighbourhood, U, of f such that if g belongs to U we have $\nu_c(g) \equiv \nu_c(f) \bmod t^n$. Thus ν_c is continuous at f.

A similar proof shows that ν_b is continuous at f.

2.2.6 LEMMA. Let f belong to $A^o_{b,c}$ and suppose that f satisfies the following properties:

(i) c and b are not periodic points; and

(ii) there exists $n > 0$ such that $f^n(c) = b$.

Then given any integer, $m \geq 0$, we can find a neighbourhood, U, of f such that if g belongs to U we have either:

(i) $\nu_c(g) \equiv \nu_c(f) \bmod t^m$; or

(ii) $\nu_c(g) \equiv \nu_c(f) - 2t^n\nu_b(f) \bmod t^m$.

Proof. Let $\nu_c(f) = \sum_0^\infty k_i t^i$. Then it is easily seen that $\nu_c(f)$ is either equal to $\sum_0^{n-1} k_i t^i + 2t^n\theta_f(b+)$ or equal to $\sum_0^{n-1} k_i t^i - 2t^n\theta_f(b-)$.

We shall prove the lemma for the case when $\nu_c(f) = \sum_0^{n-1} k_i t^i + 2t^n\theta_f(b+)$. The other case can be proved in a similar way.

Using lemma 2.2.1 we can see that for g sufficiently close to f we have either $\nu_c(g) \equiv \nu_c(f)$ modulo t^m or $\nu_c(g) \equiv \sum_0^{n-1} k_i t^i + 2t^n\theta_f(b-)$ modulo t^m.

If the first case occurs we have nothing more to prove, so suppose that

$\nu_c(g) \equiv \sum_0^{n-1} k_i t^i + 2t^n \theta_f(b-) \bmod t^m$. Then $\nu_c(f) - \nu_c(g) \equiv 2t^n[\theta_f(b+) - \theta_f(b-)] \bmod t^m$.

The proof is completed once it has been noted that

$$\theta_f(b+) - \theta_f(b-) = \nu_b(f) .$$

A similar proof gives the following lemma.

2.2.7 LEMMA. Let f belong to $\mathcal{A}^o_{b,c}$ and suppose that f satisfies the following properties:

(i) c and b are not periodic points; and

(ii) there exists $n > 0$ such that $f^n(b) = c$.

Then given any integer, $m \geq 0$ we can find a neighbourhood, U , of f such that if g belongs to U we have either:

(i) $\nu_b(g) \equiv \nu_b(f) \bmod t^m$; or

(ii) $\nu_b(g) \equiv \nu_b(f) - 2t^n \nu_c(f) \bmod t^m$.

2.2.8 DEFINITION. Let $D: \mathcal{A}^o_{b,c} \to \mathbb{Z}[[t]]$ be the map which sends a function to its kneading determinant.

2.2.9 LEMMA. Let f belong to $\mathcal{A}^o_{b,c}$. Suppose that c and b are not periodic points. Then D is continuous at f .

Proof. This is easily seen to be true after noting that the discontinuities of lemmas 2.2.6 and 2.2.7 correspond to elementary row operations of the kneading matrix and hence do not affect the determinant.

2.2.10 NOTATION. Let $\sum_0^\infty a_n t^n$ and $\sum_0^\infty b_n t^n$ belong to $\mathbb{Q}[[t]]$. Then we will write $\sum_0^\infty a_n t^n \ll \sum_0^\infty b_n t^n$ if for all n we have $|a_n| \leq |b_n|$.

2.2.11 NOTATION. Let g belong to $\mathcal{A}$ and let its kneading determinant be

$\sum_0^\infty D_i(g)t^i$. Then given a real number, s , we will write $D(g)(s)$ for $\sum_0^\infty D_i(g)s^i$, where this sum converges.

2.2.12 THEOREM. *Let* f *belong to* $\mathcal{A}^0_{b,c}$ *and have the property that both* b *and* c *are not periodic. Then the map* $\mathrm{ent}:\mathcal{A}^0_{b,c} \to \mathbb{R}$ *which maps a function to its entropy is continuous.*

Proof. Let the kneading matrix of f be

$$\begin{vmatrix} \nu_c^{I}(f) & \nu_b^{I}(f) \\ \\ \nu_c^{II}(f) & \nu_b^{II}(f) \end{vmatrix} .$$

It is clear that $\nu_c^I \ll 2(1-t)^{-1}$, $\nu_c^{II}(f) \ll 2(1-t)^{-1}$, $\nu_b^I(f) \ll 2(1-t)^{-1}$ and that $\nu_b^{II}(f) \ll 2(1-t)^{-1}$. So $D(f)$ satisfies $D(f) \ll \dfrac{8}{(1-t)^2}$.

Let $r < 1$ be a positive real number and let $\sum_0^\infty a_i t^i$ denote $\dfrac{8}{(1-t)^2}$. Then given $\varepsilon > 0$ we can find a positive integer, N , such that $\sum_N^\infty a_i r^i$ is less than $\varepsilon/2$. Since $D(g) \ll \dfrac{8}{(1-t)^2}$ for any g belonging to $\mathcal{A}$ we have $|\sum_N^\infty D_i(g)s^i| < \varepsilon$ for any real integer s with $|s| < r$.

Lemma 2.2.7 shows that we can choose a neighbourhood, U , of f such that if g belongs to U we have $D(g) \equiv D(f)$ modulo t^N . Thus $|D(g)(s)-D(f)(s)| < \varepsilon$ for any s satisfying $|s| \leq r$. Therefore $D(g)(s)$ tends uniformly to $D(f)(s)$ throughout the disc of radius r . Since the zeros of a complex analytic function vary continuously with the function we have $\mathrm{ent}(g)$ tends to $\mathrm{ent}(f)$ as g tends to f .

2.3 CONTINUITY OF ENTROPY IN $\mathcal{A}^1_{b,c}$.

2.3.0 In this section we will prove that entropy is continuous at any function belonging to $\mathcal{A}^1_{b,c}$ with respect to the C^1-topology.

2.3.1 PROPOSITION. _Let_ f _belong to_ $\mathcal{A}^1_{b,c}$ _and have the property that both_ b _and_ c _are not periodic_. _Then the map_ $\mathrm{ent}:\mathcal{A}^1_{b,c} \to \mathbb{R}$ _which maps a function to its entropy is continuous_.

Proof. This is a direct consequence of theorem 2.2.12 after noting that if U is an open subset of $\mathcal{A}^0_{b,c}$ then $U \cap \mathcal{A}^1_{b,c}$ is an open subset of $\mathcal{A}^1_{b,c}$.

2.3.2 LEMMA. _Let_ f _belong to_ $\mathcal{A}^1_{b,c}$ _and suppose that_ $f^n(b) = b$. _Then for every_ g _which is sufficiently close to_ f _in the space_ $\mathcal{A}^1_{b,c}$ _we have_ $g^n(b)-b$, $g^{2n}(b)-b$, $g^{3n}(b)-b$, ... _are either all zero or have the same sign_. _Furthermore given_ $\varepsilon > 0$ _there exists a_ $\delta > 0$ _such that_ $|g^m(b)-f^m(b)| < \varepsilon$ _uniformly for all_ m _whenever_ $||g-f|| < \delta$.

Proof. For convenience we will let F denote f^n and G denote g^n . Since the first derivative $\dot{F}$ of F is zero at b we can choose an $\varepsilon > 0$ such that $|\dot{F}(x)| < \frac{1}{4}$ for $|x-b| \leq \varepsilon$. Now choose g close enough to f so that $|\dot{G}(x)| < \frac{1}{2}$ for $|x-b| \leq \varepsilon$ and so that $|G(b)-b| < \frac{\varepsilon}{2}$.

If $G(b) = b$ then there is nothing more to prove. Suppose that $G(b) \neq b$, then the Mean Value theorem gives $|G^2(b)-G(b)| < \frac{1}{2}|G(b)-b| < \frac{\varepsilon}{2^2}$. By easy induction we can see that

$$|G^n(b)-G^{n-1}(b)| < \tfrac{1}{2}\,|G^{n-1}(b)-G^{n-2}(b)| < \frac{\varepsilon}{2^n} .$$

Thus $|G^n(b)-G(b)| < \frac{\varepsilon}{2}$ and the conclusion follows easily.

Similarly we can prove the following.

2.3.3 LEMMA. _Let_ f _belong to_ $\mathcal{A}^1_{b,c}$ _and suppose that_ $f^n(a) = a$. _Then for_

<u>any</u> g <u>which is sufficiently close to</u> f <u>in the space</u> $\mathcal{A}^1_{b,c}$ <u>we have</u> $g^n(a)-a,\ g^{2n}(a)-a,\ g^{3n}(a)-a,\dots$ <u>are either all zero or have the same sign.</u> <u>Furthermore given</u> $\varepsilon > 0$ <u>there exists a</u> $\delta > 0$ <u>such that</u> $|g^m(a)-f^m(a)| < \varepsilon$ <u>uniformly for all</u> m <u>whenever</u> $||g-f|| < \delta$.

2.3.4 LEMMA. <u>Let</u> f <u>belong to</u> $\mathcal{A}^1_{b,c}$ <u>and satisfy the following</u>

(i) b <u>is periodic with period</u> n ; <u>and</u>

(ii) <u>for all</u> $m \geq 1$, $f^m(b) \neq a$.

<u>Then we can choose a neighbourhood,</u> U , <u>of</u> f <u>such that if</u> g <u>belongs to</u> U <u>we have either</u>

(i) $\nu_b(f) = \nu_b(g)$; or

(ii) $\nu_b(g) = \dfrac{(1-t^n)}{(1+t^n)}\nu_b(f)$.

<u>Proof</u>. Let $\sum_0^\infty \nu_b^i t^i$ denote $\nu_b(f)$. Then we can see that $\nu_b(f)-\sum_0^{n-1}\nu_b^i t^i$ is equal to either $2t^n\theta_f(b+)$ or $-2t^n\theta_f(b-)$.

We will prove the lemma for the case $\nu_b(f)-\sum_0^{n-1}\nu_b^i t^i = 2t^n\theta_f(b+)$, the other case can be proved in a similar way.

From the proof of lemma 2.2.5 we can see that we can choose a neighbourhood, W , of f such that if g belongs to W we have $\nu_b(g) \equiv \nu_b(f) \bmod t^n$. Lemma 2.3.2 tells us that we can restrict W to an open neighbourhood, U , of f such that if g belongs to U we have that $\nu_b(g)$ is equal to either $\sum_0^{n-1}\nu_b^i t^i+2t^n\theta_g(b+)$ or $\sum_0^{n-1}\nu_b^i t^i+2t^n\theta_g(b-)$.

If the first case occurs we obtain $\nu_b(f)-\nu_b(g) = 2t^n(\theta_f(b+)-\theta_g(b+))$. Using lemma 1.2.6 gives $\nu_b(f)-\nu_b(g) = t^n(\nu_b(f)-\nu_b(g))$ which implies that $\nu_b(f) = \nu_b(g)$.

If the second case occurs we have $\nu_b(f)-\nu_b(g) = 2t^n(\theta_f(b+)-\theta_g(b-))$. Using lemma 1.2.6 gives $\nu_b(f)-\nu_b(g) = t^n(\nu_b(f)+\nu_b(g))$. Thus

$$\nu_b(g) = \nu_b(f)\,\frac{(1-t^n)}{(1+t^n)} \ .$$

A similar proof to the above, using lemma 2.3.3 gives the following.

2.3.5 LEMMA. Let f belong to $\mathcal{A}^1_{b,c}$ and satisfy the following

(i) c is periodic with period n ; and

(ii) for all $m \geq 1$, $f^m(c) \neq a$.

Then we can choose a neighbourhood, U , of f such that if g belongs to U we have either

(i) $\nu_c(g) = \nu_c(f)$; or

(ii) $\nu_c(g) = \dfrac{(1-t^n)}{(1+t^n)}\,\nu_c(f)$.

2.3.6 LEMMA. Let f belong to $\mathcal{A}^1_{b,c}$. Suppose there exist positive integers, p and k , such that $f^p(c) = b$ and $f^k(b) = c$. Then for g sufficiently close to f we have $\eta\times D(f) = \mu\times D(g)$, where μ and η denote smooth, non-zero analytic functions defined for $|t| < 1$.

Proof. Let $\sum_0^\infty \nu_b^i t^i$ denote $\nu_b(f)$. Then $\nu_b(f)$ is equal to either

$\sum_0^{k-1} \nu_b^i t^i + 2t^k\theta_f(c+)$ or $\sum_0^{k-1} \nu_b^i t^i - 2t^k\theta_f(c-)$. We will only give the proof for $\nu_b(f) = \sum_0^{k-1} \nu_b^i t^i + 2t^k\theta_f(c+)$, as the proof for the other case is similar.

We can choose a small neighbourhood, U , of f such that if g belongs to U either $\nu_b(g) = \sum_0^{k-1} \nu_b^i t^i + 2t^k\theta_g(c+)$ or $\nu_b(g) = \sum_0^{k-1} \nu_b^i t^i + 2t^k\theta_g(c-)$. Simple calculation similar to that in the proof of 2.3.4 shows $\nu_b(f)-\nu_b(g)$ is equal to either $t^k(\nu_c(f) + \nu_c(g))$ or $t^k(\nu_c(f) - \nu_c(g))$. Similarly we obtain that $\nu_c(f)-\nu_c(g)$ is equal to either $t^p(\nu_b(f)+\nu_b(g))$ or $t^p(\nu_b(f)-\nu_b(g))$.

Suppose that $\nu_b(f)-\nu_b(g) = t^k(\nu_c(f)-\nu_c(g))$ and that $\nu_c(f)-\nu_c(g) = t^p(\nu_b(f)-\nu_b(g))$. Solving these two simultaneous equations gives $\nu_b(f) = \nu_b(g)$. Similarly, if $\nu_b(f)-\nu_b(g) = t^k(\nu_c(f)+\nu_c(g))$ and $\nu_c(f)-\nu_c(g) = t^p(\nu_b(f)+\nu_b(g))$ we obtain $(1-t^{p+k})\nu_c(g) = (1+t^{p+k})\nu_c(f)-2t^p\nu_b(f)$. If $\nu_b(f)-\nu_b(g) = t^k(\nu_c(f)+\nu_c(g))$ and $\nu_c(f)-\nu_c(g)$ equals $t^p(\nu_b(f)-\nu_b(g))$ we obtain

$$(1+t^{p+k})\nu_b(f)-2t^k\nu_c(f) = (1+t^{p+k})\nu_b(g) \quad .$$

If $\nu_b(f)-\nu_b(g) = t^k(\nu_c(f)-\nu_c(g))$ and $\nu_c(f)-\nu_c(g)$ equals $t^p(\nu_b(f)+\nu_b(g))$ we obtain

$$(1+t^{p+k})\nu_c(g) = \nu_c(f)(1-t^{p+k}) \quad .$$

So in all cases we can find smooth analytic functions μ and η such that $\eta\times D(f) = \mu\times D(g)$.

2.3.7 THEOREM. <u>Let</u> f <u>belong to</u> $\mathcal{A}^1_{b,c}$. <u>Then the map</u> $\mathrm{ent}:\mathcal{A}^1_{b,c}\to\mathbb{R}$ <u>which maps a function to its entropy is continuous</u>. <u>Moreover</u>, <u>if both</u> b <u>and</u> c <u>are periodic points the entropy map is locally constant</u>.

<u>Proof</u>. Suppose both b and c are periodic points of f . Then we have shown that we can find a neighbourhood, U , of f such that if g belongs to U we have $\mu\times D(g) = \eta\times D(f)$ where both η and μ are analytic functions. Let r be the first zero of $D(f)$. Then, since both η and μ are non-zero for $|t| < 1$, r must be the first zero of $D(g)$.

If just one of b or c is periodic then we can repeat the argument given in 2.2.12 to complete the proof.

that $\nu_c(f)-\nu_c(g)$ is equal to either $t^p(\nu_b(f)+\nu_b(g))$ or $t^p(\nu_b(f)-\nu_b(g))$.

Suppose that $\nu_b(f)-\nu_b(g) = t^k(\nu_c(f)-\nu_c(g))$ and that $\nu_c(f)-\nu_c(g) = t^p(\nu_b(f)-\nu_b(g))$. Solving these two simultaneous equations gives $\nu_b(f) = \nu_b(g)$. Similarly, if $\nu_b(f)-\nu_b(g) = t^k(\nu_c(f)+\nu_c(g))$ and $\nu_c(f)-\nu_c(g) = t^p(\nu_b(f)+\nu_b(g))$ we obtain $(1-t^{p+k})\nu_c(g) = (1+t^{p+k})\nu_c(f)-2t^p\nu_b(f)$. If $\nu_b(f)-\nu_b(g) = t^k(\nu_c(f)+\nu_c(g))$ and $\nu_c(f)-\nu_c(g)$ equals $t^p(\nu_b(f)-\nu_b(g))$ we obtain

$$(1+t^{p+k})\nu_b(f)-2t^k\nu_c(f) = (1+t^{p+k})\nu_b(g) \quad .$$

If $\nu_b(f)-\nu_b(g) = t^k(\nu_c(f)-\nu_c(g))$ and $\nu_c(f)-\nu_c(g)$ equals $t^p(\nu_b(f)+\nu_b(g))$ we obtain

$$(1+t^{p+k})\nu_c(g) = \nu_c(f)(1-t^{p+k}) \quad .$$

So in all cases we can find smooth analytic functions μ and η such that $\eta\times D(f) = \mu\times D(g)$.

2.3.7 THEOREM. <u>Let</u> f <u>belong to</u> $\mathcal{A}^1_{b,c}$. <u>Then the map</u> ent: $\mathcal{A}^1_{b,c} \to \mathbb{R}$ <u>which maps a function to its entropy is continuous</u>. <u>Moreover</u>, <u>if both</u> b <u>and</u> c <u>are periodic points the entropy map is locally constant</u>.

<u>Proof</u>. Suppose both b and c are periodic points of f . Then we have shown that we can find a neighbourhood, U , of f such that if g belongs to U we have $\mu\times D(g) = \eta\times D(f)$ where both η and μ are analytic functions. Let r be the first zero of D(f) . Then, since both η and μ are non-zero for $|t| < 1$, r must be the first zero of D(g) .

If just one of b or c is periodic then we can repeat the argument given in 2.2.12 to complete the proof.

CHAPTER 3 ROTATION NUMBERS

3.0 INTRODUCTION

3.0.1 Let f belong to $\mathcal{A}$ and let F be a lift of f. In [8] Newhouse, Palis and Takens define the rotation number of $x \in S^1$ with respect to F to be $\rho(F,x) = \overline{\lim_{n\to\infty}} \frac{1}{n}(F^n(x)-x)$. The rotation set, R_F , is then the closure of $\{\rho(F,x) : x \in S^1\}$. If F' is another lift of f then $\rho(F,x) \equiv \rho(F',x) \pmod 1$ so the rotation sets R_F and $R_{F'}$ are equal up to translation by an integer.

In [8] it is shown that R_F is always a closed interval of $\mathbb{R}$. In fact, for $f \in \mathcal{A}$, we can always choose a lift F with $R_F \subset [0,1]$. To see this let F be a lift satisfying $0 \leq F(0) < 1$, then we have $0 \leq F(x) < 2$ for $x \in [0,1]$ and it follows that $0 \leq F^n(x) < n+1$. Thus $\rho(F,x) \in [0,1]$ and so $R_F \subset [0,1]$. We define the rotation interval of f to be this closed interval of $[0,1]$. We define the rotation number of $y \in S^1$ with respect to f , $\rho(f,y)$ to be $\rho(F, \exp^{-1}(y))$, where $\exp$ is the map from $[0,1)$ to S^1 given by

$$\exp(t) = e^{2\pi i t} .$$

Suppose that $\frac{p}{q}$ is a rational number contained in R_f then in [8] it is shown that we can find $y \in S^1$ such that y is periodic with period q and such that $\rho(f,y) = \frac{p}{q}$.

The following lemma shows that sometimes the rotation number of a point x can be calculated from the address.

3.0.2 LEMMA. Let f belong to $\mathcal{A}$. Suppose that there exists $x \in S^1$ such that x is periodic of period n ; suppose also that $f^i(x)$ is not equal to a_f or b_f for $i \geq 0$. Let $(A_0, A_1, A_2, \ldots)$ be the itinerary of x and let m be the cardinality of $\{j \in \mathbb{Z} : 0 \leq j < n$ such that $A_j \neq I\}$. Then x has rotation number $\frac{m}{n}$.

Proof. Choose a lift F of f with $0 \leq F(0) < 1$. Then it is clear that $F(\exp^{-1}(c_f)) = 1$ and $F(1) = 1+F(0)$.

Thus $F(\exp^{-1}(x)) > 1$ if x belongs to (c_f, a_f) and $F(\exp^{-1}(x)) < 1$ if x belongs to (a_f, c_f). Simple induction* then shows that $m < F^n(\exp^{-1}(x)) < m+1$, and so $m-1 < F^n(\exp^{-1}(x)) - \exp^{-1}(x) < m+1$. Since x is periodic we know that $F^n(\exp^{-1}(x)) - \exp^{-1}(x)$ is an integer. Thus $F^n(\exp^{-1}(x)) - \exp^{-1}(x) = m$ and so $\rho(f,x) = \frac{m}{n}$.

In this chapter we define a kneading invariant $\theta_{n,p}$ and in the main theorem show that if there exists a point with rotation number $\frac{n}{p}$ then there must be a periodic point with invariant coordinate $\theta_{n,p}$.

In order to prove this we need to know for which $\theta \in V[[t]]$ there exists an $x \in S^1$ with $\theta(x) = \theta$. In the first section we define an allowability condition (c.f. Leo Jonker [5]) and show that if θ satisfies this condition $\theta(x) = \theta$ for some $x \in S^1$.

Section 3.2 is a technical section involving properties of strings of positive integers under cyclic permutation. The results of this section are then used in 3.3 and 3.4 to prove the main theorem (3.4.0).

3.1 ALLOWABLE ELEMENTS

3.1.0 Recall from the first chapter that V is a vector space over the rationals with basis $\{I, II, III\}$, and that $V[[t]]$ is the $Q[[t]]$-module consisting of all formal power series with coefficients in V.

3.1.1 DEFINITION. Let ψ belong to $V[[t]]$ and let $\psi = \sum_0^\infty \psi_i t^i$. Then we will define ψ to be allowable if it satisfies the following conditions:

(i) ψ is periodic, ie. we can find an integer $n > 0$ such that for all i we have $\psi_{n+i} = \psi_i$.

(ii) For all i we have ψ_i is equal to either $+I$ or $+II$.

(iii) For all i we have

* The inductive argument uses the fact that for all $x \in R$ we have $F(x+1) = 1 + F(x)$.

$$\theta(a+) < \psi_i + \psi_{i+1}t + \ldots + \psi_{i+m}t^m \ldots < \theta(b-) .$$

3.1.2 The purpose of this section is to show that if ψ is an allowable element of $V[[t]]$ then we can find an $x \in S^1$ with $\theta(x) = \psi$ and such that x is periodic with the period of ψ.

3.1.3 LEMMA. Let ψ be an allowable element. Then either ψ is less than $\theta(c-)$ or ψ is greater than $\theta(c+)$.

Proof. Suppose that we have $\theta(c-) \leq \psi \leq \theta(c)$. Then comparing the first coefficients of the power series $\theta(c-)$, ψ and $\theta(c)$ gives $I \leq \psi_0 \leq \frac{I+II}{2}$. Therefore $\psi_0 = I$ since for ψ to be allowable ψ_0 must equal to I or II. Now if we compare the first and second coefficients of $\theta(c-)$ and ψ we obtain $I+IIIt \leq I+\psi_1 t$. But this implies that $\psi_1 = III$ which contradicts the second condition of allowability. Therefore we must have either $\psi < \theta(c+)$ or $\psi > \theta(c)$

Suppose that we have $\theta(c) \leq \psi \leq \theta(c+)$. Then comparing first coefficients gives $\frac{I+II}{2} < \psi_0 \leq II$. So we must have $\psi_0 = II$. Since $\psi \leq \theta(c+)$ we must have $II+\psi_1 t+\psi_2 t^2+\ldots \leq II+t\theta(a+)$ but this contradicts the third condition for allowability. So either ψ is less than $\theta(c-)$ or ψ is greater than $\theta(c+)$.

3.1.4 LEMMA. Let ψ be allowable and suppose there exists x belonging to S^1 with $\theta(x-) \leq \psi \leq \theta(x+)$. Then θ is continuous at x and $\theta(x) = \psi$.

Proof. If θ is continuous at x then $\theta(x-) = \theta(x) = \theta(x+)$ and we are finished. So we will suppose that θ is not continuous at x. Lemma 1.3.4 tells us that for some $n \geq 0$ we must have

$$\theta(x) = \psi_0 + \psi_1 t + \ldots + \psi_{n-1}t^{n-1} + t^n\theta(f^n(x)) ,$$

$$\theta(x+) = \psi_0 + \psi_1 t + \ldots + \psi_{n-1}t^{n-1} + t^n\theta(f^n(x)+) ,$$

and $$\theta(x-) = \psi_0 + \psi_1 t + \ldots + \psi_{n-1}t^{n-1} + t^n\theta(f^n(x)-) ,$$

where $f^n(x)$ is equal to either b or c.

This implies that $\theta(f^n(x)-) \leq \psi_n+\psi_{n+1}t+\ldots\leq \theta(f^n(x)+)$. We cannot have $f^n(x) = b$ because this would contradict the third condition of allowability. But $f^n(x)$ cannot equal c by Lemma 3.1.3. So θ must be continuous at x .

3.1.5 LEMMA. Let ψ be allowable. Then we can find an x belonging to S^1 with $\theta(x) = \psi$.

Proof. Let $x = \sup\{y|\theta(y) < \psi\}$. Since $\theta(a-) < \psi$ this supremum does exist. Then it is clear that $\theta(x-) \leq \psi \leq \theta(x+)$ and thus by Lemma 3.1.4 we have $\theta(x) = \psi$.

3.1.6 PROPOSITION. Let ψ be allowable and have period n . Then we can find y belonging to S^1 such that $\theta(y) = \psi$ and such that y is periodic with period n .

Proof. The previous lemma tells us that we can find a point x belonging to S^1 with $\theta(x) = \psi$. Let K be the closure of the convex hull of $\{f^{kn}(x)|k\varepsilon N\}$. Then clearly K is a closed interval and $f^n:K \to K$ is an orientation preserving homeomorphism. Let K be denoted by $[X,Y]$, then $\theta(Y-) = \psi$ and so by Lemma 3.1.4 we must have $\theta(Y) = \psi$. Similarly $\theta(X) = \psi$, and so $f^n:K \to K$ is onto. Thus $f^n(X) = X$, $f^n(Y) = Y$, and X and Y cannot have period less than n , as this would imply that ψ had period less than n .

3.2 (n,p)-ORBITS

3.2.0 Throughout this section let n and p denote two coprime positive integers with $p > n$.

3.2.1 DEFINITION. An (n,p)-element is a string of n positive integers, $a_0a_1\ldots a_{n-1}$ such that $\sum_0^{n-1} a_i = p$.

We shall let $S_{n,p}$ denote the set of (n,p)-elements and order elements of $S_{n,p}$ with the usual lexicographical ordering.

3.2.2 Given an (n,p)-element, $X = a_0 \ldots a_{n-1}$,

let $\sigma(X) = a_1 a_2 \ldots a_{n-1} a_0$, $\sigma^2(X) = a_2 a_3 \ldots a_{n-1} a_0 a_1, \ldots$ $\sigma^i(X) = a_i a_{i+1} \ldots a_{i-1}$.

3.2.3 DEFINITION. The <u>orbit</u> of X is $\{\sigma^i(X) | 0 \leq i \leq n-1\}$, and is denoted $[X]$.

3.2.4 DEFINITION. Given an orbit $[X]$, let $[X]{\downarrow}$ denote the smallest element in $[X]$. i.e. $[X]{\downarrow} = b_0 \ldots b_{n-1}$ where $b_0 .. b_{n-1}$ is an element of $[X]$ and if $c_0 .. c_{n-1} \varepsilon [X]$ then $b_0 .. b_{n-1} \leq c_0 .. c_{n-1}$.

3.2.5 DEFINITION. Let $[X]{\uparrow}$ denote the largest element in $[X]$.

3.2.6 DEFINITION. The <u>greatest smallest</u> (n,p)-<u>element</u>, denoted $gs(n,p)$, is $\max\{[X]{\downarrow} \mid X \varepsilon S_{n,p}\}$.

3.2.7 DEFINITION. The <u>least largest</u> (n,p)-<u>element</u>, denoted $\ell\ell(n,p)$, is $\min\{[X]{\uparrow} \mid X \varepsilon S_{n,p}\}$.

3.2.8 EXAMPLE. It is easily seen that $S_{3,5} = \{113,131,113,122,212,221\}$. So we have two orbits $[113]$ and $[122]$. Now $[113]{\uparrow} = 311$ and $[122]{\uparrow} = 221$. Therefore $\ell\ell(3,5) = 221$. Similarly it is easy to check that $gs(3,5) = 122$.

3.2.9 The aims of this section are to show that $\ell\ell(n,p)$ and $gs(n,p)$ belong to the same orbit and to give an explicit way of calculating this orbit. We now fix the integers p and n.

3.2.10 Let Z denote the unique integer such that $Z < \frac{p}{n} < Z+1$.

3.2.11 LEMMA. <u>Let</u> $gs(n,p) = b_0 .. b_{n-1}$. <u>Then for all</u> i <u>we have</u> $b_i \geq Z$.

<u>Proof</u>. It is clear that the integer $p-Zn$ lies between 0 and n . We will consider $c_0 \ldots c_{n-1}$ where $c_i = \begin{cases} Z & \text{if } 0 \leq i < n-(p-Zn)-1 \\ Z+1 & \text{if } n-(p-Zn) \leq i \leq n-1 \end{cases}$.

It is easily seen that $c_0 \ldots c_{n-1}$ belongs to $S_{n,p}$ and that $c_0 \ldots c_{n-1} = [c_0 \ldots c_{n-1}]\downarrow$. Thus we must have $gs(n,p) \geq c_0 \ldots c_{n-1}$.

Now suppose for a contradiction that there exists b_i with $b_i < Z$. Then since $b_0 \ldots b_{n-1} = [b_0 \ldots b_{n-1}]\downarrow$ we must have $b_0 < Z$, but this would imply that $b_0 \ldots b_{n-1} < c_0 \ldots c_{n-1}$ which contradicts the fact that $gs(n,p) \geq c_0 \ldots c_{n-1}$.

3.2.12 LEMMA. Let $\ell\ell(n,p) = d_0 \ldots d_{n-1}$. Then for all i we have $d_i \leq Z+1$.

Proof. The proof is similar to the proof of 3.2.11.

3.2.13 LEMMA. Let $gs(n,p) = b_0 \ldots b_{n-1}$. Then $b_i \leq Z+1$ for all i .

Proof. Suppose that $b_m > Z+1$ for some m . Let k be the smallest integer such that $b_k > Z+1$. Since $b_0 \ldots b_{n-1} = [b_0 .. b_{n-1}]\downarrow$ we know that $b_0 .. b_k \leq b_i \ldots b_{i+k}$, where the subscripts are taken modulo n .

Consider the following set of integers, $S = \{j \mid b_0 \ldots b_{k-1} = b_j \ldots b_{j+k-1}\}$. S is non-empty since 0 belongs to S . If j_1 and j_2 belong to S with $j_1 < j_2$ then notice that we must have $j_2 + k < n$ and $j_1 + k < j_2$.

We shall now define a new (n,p) element.

Let $c_0 .. c_{n-1}$ be given by $c_i = \begin{cases} b_i+1 & \text{if } i = j+k-1 \text{ for some } j \in S \\ b_i-1 & \text{if } i = j+k \text{ for some } j \in S \\ b_i & \text{otherwise.} \end{cases}$

It is easy to see that for all i we have $c_i c_{i+1} \ldots c_{i+k} > b_0 \ldots b_k$, where the subscripts are taken modulo n . Thus $[c_0 \ldots c_{n-1}]\downarrow$ is greater than $b_0 \ldots b_{n-1}$, but this contradicts the fact that $b_0 .. b_{n-1}$ is $gs(n,p)$. So we must have $b_i \leq Z+1$ for all i .

3.2.14 LEMMA. <u>Let</u> $\ell\ell(n,p) = d_0..d_{n-1}$. <u>Then</u> $d_i \geq Z$ <u>for all</u> i .

<u>Proof</u>. The proof is similar to the proof above.

3.2.15 The previous four lemmas tell us that in order to calculate $gs(n,p)$ and $\ell\ell(n,p)$ we need only look at (n,p)-elements consisting of strings of the form $b_0..b_{n-1}$ where $b_i = Z$ or $Z+1$.

We shall now give a method of constructing such a string.

3.2.16 DEFINITION. Let $n;p$ be the string of positive integers, $a_0a_1...a_m$ defined by the following:

a_0 is given by $p = a_0n+r_0$ where $0 \leq r_0 < n$,

a_1 is given by $p+r_0 = a_1n+r_1$ where $0 \leq r_1 < n$,

a_i is given by $p+r_{i-1} = a_in+r_i$ where $0 \leq r_i < n$,

and m is the first integer such that $p+r_{m-1} = a_mn$.

3.2.17 EXAMPLE. 2;5 is equal to 23, and 3;5 is equal to 122.

3.2.18 LEMMA. i) $n;p$ <u>gives a finite sequence of integers</u>,

ii) $m = n-1$,

iii) <u>for every</u> i a_i <u>is equal either to</u> a_0 <u>or</u> $a_0 + 1$,

iv) $\sum\limits_0^m a_i = p$.

<u>Proof</u>. Since $p = a_0n+r_0$ and $p+r_0 = a_1n+r_1$ we can see that $r_1 \equiv 2r_0 \bmod n$. In general we have $r_k \equiv (k+1)r_0 \bmod n$. Thus $r_{n-1} \equiv 0 \bmod n$ and so $n;p$ must be a finite string of integers.

Suppose that there exists a k smaller than n with $kr_0 \equiv 0 \bmod n$. Then r_0 and n must have a common factor, but this implies that p and n have a common factor as $p = a_0n+r_0$, which contradicts the fact that p and n

are coprime. Thus we have shown that $m = n-1$.

Let k be an integer satisfying $0 \leq k \leq n-1$. Then, clearly $a_k \geq a_0$. By definition $r_k < n$, so $p+r_k < p+n$. Using the fact that $p = a_0n+r_0$ we obtain $p+r_k < (a_0+1)\,n+r_1$, so $a_k \leq a_0+1$. Thus a_k must equal either a_0 or a_0+1 .

We know that $p = a_0n+r_0$, $p+r_0 = a_1n+r_1,\ldots,$ $p+r_{n-2} = a_{n-1}n$. Adding up the left and right-hand sides of these equations gives $np + \sum_0^{n-2} r_i = n\sum_0^{n-1} a_i + \sum_{i=1}^{n-2} r_i$, which implies that $p = \sum_0^{n-1} a_i$.

3.2.19 LEMMA. <u>Let</u> $n;p = a_0\ldots a_{n-1}$, <u>and let</u> j <u>be any integer belonging to</u> $\{1,\ldots,n\}$. <u>Then</u> $\sum_0^{j-1} a_i$ <u>satisfies</u> $n(\sum_0^{j-1} a_i) \leq jp < n(\sum_0^{j-1} a_i + 1)$.

<u>Proof</u>. We know $p = a_0n+r_0$, $p+r_0 = a_1n+r_1,\ldots,$ $p+r_{j-2} = a_{j-1}n + r_{j-1}$. Adding up the left and right hand sides gives $jp + \sum_0^{j-2} r_i = n\sum_0^{j-1} a_i + \sum_0^{j-1} r_i$. Thus jp equals $n\sum_0^{j-1} a_i + r_{j-1}$. Using the fact that $0 \leq r_{j-1} < n$ gives $n\sum_0^{j-1} a_i \leq jp < n(\sum_0^{j-1} a_i + 1)$.

3.2.20 DEFINITION. Let $n;p = a_0\ldots a_{n-1}$ and let α,β be two formal symbols. Then we define $\psi(n;p)(\alpha,\beta)$ to be the string of symbols α and β given by the following. $\psi(n;p)(\alpha,\beta) = \gamma_0\ldots\gamma_{p-1}$,

$$\text{where } \gamma_j = \begin{cases} \alpha & \text{if } j \equiv \sum_0^i a_r \pmod{p} \text{ for some } i \text{ satisfying } 0 \leq i \leq n-1 \\ \beta & \text{otherwise.} \end{cases}$$

3.2.21 EXAMPLE. We know that $3;5 = 122$. Thus it is easily seen that $\psi(3;5)(\alpha,\beta) = \alpha\alpha\beta\alpha\beta$. Now $1;3 = 3$ so $\psi(1;3)(\alpha,\beta) = \alpha\beta\beta$. Notice that $\psi(1;3)(1,2) = 122 = 3;5$.

3.2.22 LEMMA. Let $p = a_0n+r_0$. Then $n;p = \psi(n-r_0,n)(a_0,a_0+1)$.

Proof. Let $n;p = a_0 \dots a_{n-1}$. Then it is easily seen that $a_k = a_0 + 1$ if and only if $kr_0 < jn \leq (k+1)r_0$ for some positive integer j . Similarly $a_k = a_0$ if and only if $k(n-r_0) \leq jn < (k+1)(n-r_0)$ for some positive integer j.

Let ℓ_1, denote the first i such that $a_i = a_0$, and ℓ_j denote the j^{th} i such that $a_i = a_0$. Then we can see that ℓ_j, satisfies $\ell_j(n-r_0) \leq (j-1)n < (\ell_j+1)(n-r_0)$.

Let $n-r_0;n = c_0 \dots c_{n-r_0}$. Then Lemma 3.2.19 tells us that for j belonging to $\{2,3,\dots,n-r\}$ we have $\ell_j = \sum_0^{j-2} c_i$, and $\ell_1 \equiv 0 \equiv n \equiv \sum_0^{n-r_0} c_i (\mathrm{mod}\ n)$. So the two sets, $\{\ell_i \mid 1 \leq i \leq n-r_0\}$ and $\{\sum_0^{r_0} c_i (\mathrm{mod}\ n) \mid 0 \leq r_0 \leq n-r_0-1\}$ are equal. Thus we have $n;p = \psi(n-r_0;n)(a_0,a_0+1)$.

3.2.23 COROLLARY. Let $p = a_0n+r_0$. Then $\psi(n;p)(c,d)$ equals $\psi(n-r_0;n)[\psi(1;a_0)(c,d),\ \psi(1;a_0+1)(c,d)]$.

Proof. This follows immediately from 3.2.22.

3.2.24 PROPOSITION. $gs(n,p) = n;p$.

Proof. We will prove this by induction on p .

If $p = 2$ then we must have $n = 1$ and so $gs(1,2) = 2$ which equals $1;2$.

If $p = 3$ and $n = 1$ then we have $gs(1,3) = 3$ which equals $1;3$. If $p = 3$ and $n = 2$ then it is easily seen that $gs(2,3) = 12$ which equals $2;3$.

So the proposition is true for $p = 2$ and $p = 3$. Suppose we have proved it for all positive integers, p , less than q . We will now show that $gs(n,q)$ equals $n;q$.

Let $gs(n,q) = a_0 \dots a_{n-1}$. Then 3.2.15 tells us that for all i , a_i is equal to either a_0 or a_0+1 . So we shall consider only (n,q)-elements $b_0..b_{n-1}$ having the properties that, for all i , b_i is either equal to a_0 or a_0+1 and such that $b_0 = a_0$.

Given any such string $B = b_0..b_{n-1}$ we will define substrings $A_0^B, A_1^B \dots A_{n-r_0-1}^B$ in the following way.

$A_0^B = b_0 \dots b_{k_1}$ where k_1+1 is the first integer greater than zero with b_{k_1+1} equal to a_0 . $A_1^B = b_{k_1+1} \dots b_{k_2}$ where k_2+1 is the first integer greater than k_1+1 with $b_{k_2+1} = a_0$. Similarly $A_i^B = b_{k_i+1} \dots b_{k_{i+1}}$ where $k_{i+1}+1$ is the first integer greater than k_i+1 with $b_{k_{i+1}+1}$ equal to a_0 .

For example if $B = 232233$ we have

$$A_0^B = 23, \quad A_1^B = 2 \quad \text{and} \quad A_2^B = 233 .$$

Let ℓ_i^B be the length of A_i^B i.e. $k_{i+1}-k_i$. Notice that we have the following.

1) ℓ_i^B determines A_i^B

2) $\sum_0^{n-r-1} \ell_i^B = n$

3) Let c be another such (n,p)-element. Then $A_0^B \dots A_{n-r-1}^B < A_0^C \dots A_{n-r-1}^C$ if and only if $\ell_0^B \dots \ell_{n-r-1}^B < \ell_0^C \dots \ell_{n-r-1}^C$.

So it follows that $gs(n,p)$ is determined by $gs(n-r,n)$. By the inductive hypothesis we know that $gs(n-r,n) = n-r;n$.

Let $n = d(n-r)+r^1$ where $0 \leq r^1 < n-r$. Then 3.2.22 tells us that $n-r;n = \psi(n-r-r^1;n-r)(d,d+1)$.

So $gs(n,p) = \psi(n-r-r^1;n-r)[\psi(1;d)(a_0,a_0+1) , \psi(1;d+1)(a_0,a_0+1)]$

which by 3.2.23 is equal to $\psi(n-r;n)(a_0;a_0+1)$. Using 3.2.22 then tells us that this is equal to $n;q$.

3.2.25 PROPOSITION. $\ell\ell(n,p) = \psi(r,n)(a_0+1,a_0)$.

Proof. The proof follows along the lines of the proof of 3.2.25.

3.2.26 Now we have shown how to calculate $\ell\ell(n,p)$ and $gs(n,p)$ we want to show that they belong to the same (n,p) orbit.

3.2.27 LEMMA. Let $n;p = a_0 \dots a_{n-1}$. Then $a_i = a_{n-1-i}$ for $1 \leq i \leq n-1$.

Proof. Let $p = a_0 n+r$. Suppose that for some k with $k \neq 0$ and $k \neq n$ we have $a_k = a_0+1$. This means that we can find a positive integer ℓ with $kr < \ell n < (k+1)r$, but this is true if and only if $(n-k-1)r < (r-\ell)n < (n-k)r$, which implies that $a_{n-1-k} = a_0+1$.

3.2.28 DEFINITION. Given any string of symbols $X = a_0 \dots a_{n-1}$ let X^{-1} be the string of symbols $a_{n-1}a_{n-2} \dots a_1 a_0$.

3.2.29 We know that $gs(n,p) = n;p$. In order to show that $gs(n,p)$ and $\ell\ell(n,p)$ belong to the same orbit, we will show first that $\ell\ell(n,p) = (n|p)^{-1}$ and then show that $n;p$ and $(n;p)^{-1}$ belong to the same orbit.

3.2.30 LEMMA. $\psi(n;p)(\alpha,\beta) = [\psi(p-n;p)(\beta,\alpha)]^{-1}$.

Proof. We will show that $\psi(n;p)(1,2)$ equals $[\psi(p-n;p)(2,1)]^{-1}$. Notice that 3.2.22 tells us that $\psi(n;p)(1,2) = p;2p-n$ and $\psi(p-n;p)(1,2) = p;p+n$.

Let $p;2p-n = a_0 \dots a_{p-1}$ and $p;p+n = b_0 \dots b_{p-1}$. Then given an integer k satisfying $0 < k < p-1$ we have $a_k = 1$ if and only if there exists an integer $j > 0$ with $k(p-n) < jp < (k+1)(p-n)$, and $a_k = 2$ if and only if there exists an integer $j > 0$ with $kn < jp < (k+1)n$. Similarly, $b_k = 1$ if and only if there exists a $j > 0$ with $kn < jp < (k+1)n$, and $b_k = 2$ if and only if there exists an integer $j > 0$ with $k(p-n) < jp < (k+1)(p-n)$.

So it is easily seen that $a_k \neq b_k$ for $0 < k < p-1$. Therefore by 3.2.27 $a_k \neq b_{p-1-k}$ for $0 < k < p-1$. Since $a_0 = b_0 = 1$ and $a_{p-1} = b_{p-1} = 2$ we have $a_k \neq b_{p-1-k}$ for $0 \leq k \leq p-1$.

Consider $c_0 \ldots c_{p-1}$ defined by $c_i = \begin{cases} 1 & \text{if } b_i = 2 \\ 2 & \text{if } b_i = 1 \end{cases}$. It is clear that $a_k = c_{p-1-k}$ for any k satisfying $0 \leq k \leq p-1$. It is also clear that $c_0 \ldots c_{p-1} = \psi(p-n;p)(2,1)$. So we have proved that $\psi(n;p)(1,2) = [\psi(p-n;p)(2,1)]^{-1}$

3.2.31 COROLLARY. $\ell\ell(n,p) = (n;p)^{-1}$.

Proof. We know from Proposition 3.2.26 that $\ell\ell(n,p) = \psi(r;n)(a_0+1,a_0)$. The previous lemma then shows that $\ell\ell(n,p) = [\psi(n-r;n)(a_0,a_0+1)]^{-1}$ and then lemma 3.2.22 completes the proof.

3.2.32 LEMMA. $(n;p)^{-1}$ belongs to $[n;p]$.

Proof. We will prove this by induction on p . If $p = 2$ and $n = 1$ we have $1;2 = 2$ which equals $(1;2)^{-1}$. Similarly if $p = 3$ and $n = 1$ we obtain $(1;3) = (1;3)^{-1}$. If $p = 3$ and $n = 2$ then $2;3 = 12$ so $(2;3)^{-1} = 21$ which belongs to [12].

So suppose we have proved the lemma for all p less than q . We will now show that $(n;q)^{-1}$ belongs to $[n;q]$.

Let $n;q = a_0 \ldots a_{n-1}$ and $q = a_0 n+r$. Let $n-r;n = b_0 \ldots b_{n-r-1}$. Then we know from the proof of 3.2.22 that $a_j = a_0$ if and only if $j \equiv \sum_0^k b_i \pmod n$ for some k satisfying $0 \leq k \leq n-r-1$.

there exists an integer $j > 0$ with $kn < jp < (k+1)n$. Similarly, $b_k = 1$ if and only if there exists a $j > 0$ with $kn < jp < (k+1)n$, and $b_k = 2$ if and only if there exists an integer $j > 0$ with $k(p-n) < jp < (k+1)(p-n)$.

So it is easily seen that $a_k \neq b_k$ for $0 < k < p-1$. Therefore by 3.2.27 $a_k \neq b_{p-1-k}$ for $0 < k < p-1$. Since $a_0 = b_0 = 1$ and $a_{p-1} = b_{p-1} = 2$ we have $a_k \neq b_{p-1-k}$ for $0 \leq k \leq p-1$.

Consider $c_0 \ldots c_{p-1}$ defined by $c_i = \begin{cases} 1 & \text{if } b_i = 2 \\ 2 & \text{if } b_i = 1 \end{cases}$. It is clear that $a_k = c_{p-1-k}$ for any k satisfying $0 \leq k \leq p-1$. It is also clear that $c_0 \ldots c_{p-1} = \psi(p-n;p)(2,1)$. So we have proved that $\psi(n;p)(1,2) = [\psi(p-n;p)(2,1)]^{-}$

3.2.31 COROLLARY. $\ell\ell(n,p) = (n;p)^{-1}$.

Proof. We know from Proposition 3.2.26 that $\ell\ell(n,p) = \psi(r;n)(a_0+1,a_0)$. The previous lemma then shows that $\ell\ell(n,p) = [\psi(n-r;n)(a_0,a_0+1)]^{-1}$ and then lemma 3.2.22 completes the proof.

3.2.32 LEMMA. $(n;p)^{-1}$ belongs to $[n;p]$.

Proof. We will prove this by induction on p . If $p = 2$ and $n = 1$ we have $1;2 = 2$ which equals $(1;2)^{-1}$. Similarly if $p = 3$ and $n = 1$ we obtain $(1;3) = (1;3)^{-1}$. If $p = 3$ and $n = 2$ then $2;3 = 12$ so $(2;3)^{-1} = 21$ which belongs to [12].

So suppose we have proved the lemma for all p less than q . We will now show that $(n;q)^{-1}$ belongs to $[n;q]$.

Let $n;q = a_0 \ldots a_{n-1}$ and $q = a_0 n+r$. Let $n-r;n = b_0 \ldots b_{n-r-1}$. Then we know from the proof of 3.2.22 that $a_j = a_0$ if and only if $j \equiv \sum_0^k b_i \pmod n$ for some k satisfying $0 \leq k \leq n-r-1$.

Let $c_0 \ldots c_{n-1}$ be defined by

$$c_i = \begin{cases} a_0 & \text{if } i \equiv \sum_1^k b_{n-r-1} \pmod n \text{ where } 0 \leq k \leq n-r-1 \\ a_0+1 & \text{otherwise.} \end{cases}$$

Using the fact that by the inductive hypothesis we know that $b_{n-r-1} \ldots b_0$ belongs to $[b_0 \ldots b_{n-r-1}]$ it is easily checked that $c_0 \ldots c_{n-1}$ belongs to $[a_0 \ldots a_{n-1}]$.

We will now show that $c_1 c_2 \ldots c_{n-1} c_0 = a_{n-1} a_{n-2} \ldots a_1 a_0$. Let j be an integer satisfying $0 < j \leq n-r-1$. Then $c_j = a_0$ if and only if $j = \sum_1^k b_{n-r-i}$. If $j = \sum_1^k b_{n-r-i}$ it is easily seen that $n-j = \sum_0^{n-r-k} b_i$ which is the condition for a_{n-j} to equal a_0 . Thus for j satisfying $0 < j \leq n-r-1$ $c_j = a_0$ if and only if $a_{n-j} = a_0$ so $c_1 c_2 \ldots c_{n-1} = a_{n-1} a_{n-2} \ldots a_1$. Since both c_0 and a_0 equal a_0 we have $c_1 c_2 \ldots c_{n-1} c_0 = a_{n-1} a_{n-2} \ldots a_1 a_0$. Since $c_1 c_2 \ldots c_{n-1} c_0$ bleongs to $[c_0 \ldots c_{n-1}] = [n;p]$ we must have that $(n;p)^{-1}$ belongs to $[n;p]$.

3.2.33 SUMMARY. In this section we have shown that $gs(n,p) = n;p$, $\ell\ell(n,p) = (n;p)^{-1}$ and that $gs(n,p)$ and $\ell\ell(n,p)$ belong to the same orbit.

3.3 DEFINITION AND SOME PROPERTIES OF $\theta_{n,p}$.

3.3.1 DEFINITION. Let n and p be two coprime integers satisfying $0<n<p$, let $p;p+n = a_0 \ldots a_{p-1}$. Then we define an element $\theta_{n,p}$ belonging to $V[[t]]$ in the following way. $\theta_{n,p} = (\theta_0 + \theta_1 t + \ldots + \theta_{p-1} t^{p-1})(1-t^p)^{-1}$

where $\theta_i = \begin{cases} \text{I} & \text{if } a_i = 1 \\ \text{II} & \text{if } a_i = 2 \end{cases}$.

3.3.2 EXAMPLE. $4;7 = 1222$ so $\theta_{3,4} = (I+IIt+IIt^2+IIt^3)(1-t^4)^{-1}$.

3.3.3 DEFINITION. Let $\theta \in V[[t]]$ be such that θ is periodic with period p . Then we define $\uparrow\theta$ to be the maximum element of $\{\theta_0+\theta_1t+\ldots,\theta_1+\theta_2t+\theta_3t^2+\ldots \theta_2+\theta_3t+\theta_4t^2+\ldots,\ldots,\theta_{p-1}+\theta_pt+\ldots\}$, and we define $\downarrow\theta$ to be the minimum element of this set.

The aim of this section is to prove the following.

3.3.4 PROPOSITION. Let f belong to $\mathcal{A}$ and suppose that there exists $x \in S^1$ such that x is periodic with rotation number $\frac{n}{p}$ and such that $f^i(x) \in [a,b]$ for all $i \geq 0$. Then one of the following conditions is satisfied.

i) There exists a periodic point, y , with $\theta(y) = \theta_{n,p}$.

ii) There exists $i \geq 0$ with $f^i(x) = a$ and $\theta(a+) = \downarrow\theta_{n,p}$.

iii) There exists $i \geq 0$ with $f^i(x) = b$ and $\theta(b-) = \uparrow\theta_{n,p}$.

3.3.5 LEMMA. Let f belong to $\mathcal{A}$.

1) Suppose that b is periodic with period n , and that $f^i(b) \in [a,b]$ for all $i \geq 0$. Then $\theta(b-)$ is a periodic kneading invariant of period n.

2) Suppose that a is periodic with period n , and that $f^i(a) \in [a,b]$ for all $i \geq 0$. Then $\theta(a+)$ is a periodic kneading invariant of period n .

Proof. This follows easily from the fact that f restricted to $[a,b]$ is orientation preserving.

3.3.6 LEMMA. Let f belong to $\mathcal{A}$. Suppose that there exists $x \in S^1$ such that x is periodic with rotation number $\frac{n}{p}$ and such that $f^i(x) \in (a,b)$ for all $i \geq 0$. Then $\downarrow\theta(x) \leq \downarrow\theta_{n,p} \leq \uparrow\theta_{n,p} \leq \uparrow\theta(x)$.

Proof. Let $\theta(x) = \theta_0+\theta_1 t+\theta_2 t^2+\ldots+\theta_n t^n\ldots$. Then we have a natural $(p,p+n)$-element, $a_0\ldots a_{p+n-1}$, associated to $\theta(x)$ defined in the following way.

$$a_i = \begin{cases} 1 & \text{if } \theta_i = \mathrm{I} \\ 2 & \text{if } \theta_i = \mathrm{II} \end{cases}$$, this is a $(p.p+n)$ element by Lemma 3.0.2.

The proof is then an easy consequence of 3.2.33.

3.3.7 Proof of Proposition 3.3.4.

If $\theta(f^i(x)) = \theta_{n,p}$ for some $i \geq 0$ then there is nothing to prove. So we will suppose that for all $i \geq 0$, $\theta(f^i(x))$ is not equal to $\theta_{n,p}$. We will divide the proof into various cases.

Case 1. Suppose that for all $i \geq 0$ $f^i(x)$ is not equal to either a or b. Then 3.3.6 tells us that $\downarrow\theta(x) < \downarrow\theta_{n,p} \leq \uparrow\theta_{n,p} < \theta(x)$. Since $\downarrow\theta(x) \geq \theta(a-)$ and $\uparrow\theta(x) \leq \theta(a+)$ we have $\theta(a+) < \downarrow\theta_{n,p} \leq \uparrow\theta_{n,p} < \theta(b-)$. Thus $\theta_{n,p}$ is allowable (3.1.2) and by Proposition 3.1.7 there must exist a periodic point, $y \in S^1$, with $\theta(y) = \theta_{n,p}$.

Case 2. Suppose there exists an $i \geq 0$ such that $f^i(x) = a$. Then 3.3.5 implies that $\theta(a+)$ is periodic. If $\theta(a+) = \downarrow\theta_{n,p}$ we are finished. If $\theta(a+)$ is not equal to $\downarrow\theta_{n,p}$ then by an argument similar to the proof of 3.3.6 we must have $\theta(a+) < \downarrow\theta_{n,p} \leq \uparrow\theta_{n,p} < \theta(b-)$. As in case 1 this implies there exists a periodic point, y, with $\theta(y) = \theta_{n,p}$.

Case 3. The final case we have to consider is if there exists an $i \geq 0$ such that

$f^i(x) = b$. The proof of which follows the same lines as the proof for case 2.

3.4 PROOF OF THE MAIN THEOREM.

In this section we will prove the following.

3.4.0 THEOREM. *Let* f *belong to* $\mathcal{A}$. *Suppose that there exists* $x \in S^1$ *such that* x *is periodic with rotation number* $\frac{n}{p}$. *Then one of the following conditions must be satisfied.*

1) *There exists a periodic point* $y \in S^1$ *with* $\theta(y) = \theta_{n,p}$.

2) *There exists a positive integer,* i , *such that* $f^i(x) = a$ *and* $\theta(a+) = +\theta_{n,p}$.

3) *There exists a positive integer,* i , *such that* $f^i(x) = b$ *and* $\theta(b-) = +\theta_{n,p}$.

3.4.1 Let f belong to $\mathcal{A}$. We will say that $x \in S^1$ satisfies condition $B_{n/p}$ if it has the following properties.

i) x is periodic with rotation number $\frac{n}{p}$ and period p .

ii) There exists an integer $i \geq 0$ such that the address of $f^i(x)$ is III.

We know from 3.3.4 that the theorem is true if $f^i(x)$ belongs to $[a,b]$ for all $i \geq 0$. So to prove the theorem it is sufficient to prove the following.

3.4.2 PROPOSITION. *Let* f *belong to* $\mathcal{A}$ *and suppose there exists* $x \in S^1$ *satisfying condition* $B_{n/p}$. *Then* $\theta_{n,p}$ *is allowable.*

3.4.3 DEFINITION. Let $\theta = III+\theta_1 t+\theta_2 t^2+\ldots+\theta_n t^n+\ldots$ be an element of $V[[t]]$. Then we define ${}_I\theta$ and ${}_{II}\theta$ to be the following elements of $V[[t]]$.

$_I\theta = I-\theta_1 t-\theta_2 t^2 \ldots -\theta_n t^n - \ldots$ and

$_{II}\theta = II-\theta_1 t-\theta_2 t^2 \ldots -\theta_n t^n - \ldots$.

3.4.4 LEMMA. *Let* x *belong to* S^1 *with* $\theta(x) = III+\theta_1 t+\ldots$. *Then there exists* $y \in S^1$ *with* $\theta(y) = {_I\theta(x)}$, *and there exists* $z \in S^1$ *with* $\theta(z) = {_{II}\theta(x)}$.

Proof. From the fact that f is of degree one it is easily seen that there exists $y \in (a,c)$ with $f(y) = f(x)$, and that there exists $z \in (c,b)$ with $f(z) = f(x)$. Clearly $\theta(y)$ and $\theta(z)$ satisfy the lemma.

3.4.5 LEMMA. *Let* f *belong to* $\mathcal{A}$ *and suppose that there exists* $x \in S^1$ *satisfying condition* $B_{n/p}$. *Then* $\dagger\theta_{n,p} < \theta(b-)$.

Proof. Let the itinerary of x be $(A_0,A_1,\ldots A_{p-1},A_0,A_1,\ldots)$.
Let $B^* = (B_0,B_1,\ldots B_{p-1},B_0,B_1,\ldots)$

where $B_i = \begin{cases} A_i & \text{if } A_i \text{ not equal to III} \\ II & \text{if } A_i \text{ equals III} \end{cases}$. Then B^* defines $\theta \in V[[t]]$

in the obvious way. It is easily checked that $\theta < \theta(x)$ and that if $A(x) = III$ we have $\theta <_{II}\theta(x)$. We can now see that we must have $\dagger\theta < \theta(b-)$, but by section 3.2 we know that $\dagger\theta_{n,p} \leq \dagger\theta$, so $\dagger\theta_{n,p} < \theta(b-)$.

3.4.6 To prove proposition 3.4.2 and hence the theorem we only need to show that $\dagger\theta_{n,p} > \theta(a-)$. It will suffice to prove the following.

3.4.7 LEMMA. *Let* f *belong to* $\mathcal{A}$ *and suppose there exists* $x \in S^1$ *satisfying condition* $B_{n/p}$. *Then one of the following conditions must be satisfied*.

1) There exists an integer, $i \geq 0$, such that $\theta(f^i(x)) < \dagger\theta_{n,p}$.

2) There exists an integer, $i \geq 0$, such that $A(f^i(x)) = III$ and ${}_I\theta(f^i(x)) < +\theta_{n,p}$.

3.4.8 DEFINITION. Let x belong to S^1 . Suppose that $A(x) = I$ and that x is periodic with period p and rotation number $\frac{n}{p}$. Let the itinerary of x be $(A_0, A_1, \ldots A_{p-1}, \ldots)$. Then we define the substrings $B_0, B_1, \ldots, B_{p-n-1}$ by the following.

$B_0 = A_0 \ldots A_{k_1}$ where $A_i \neq I$ for $0 < i \leq k_1$ and $A_{k_1+1} = I$,

$B_1 = A_{k_1+1} \ldots A_{k_2}$ where $A_i \neq I$ for $k_1+1 < i \leq k_2$ and $A_{k_2+1} = I$,

$\ldots,\ B_{p-n-1} = A_{k_{p-n-1}+1} \ldots A_{p-1}$.

3.4.9 DEFINITION. The length of B_i , denoted $\ell(B_i)$ is $k_{i+1}+1-k_i$.

3.4.10 EXAMPLE. Let x be periodic with period 7 and suppose that $A^*(x) = (I,II,III,I,I,II,II,\ldots.)$. Then $B_0 = I\ II\ III$, $B_1 = I$, $B_2 = I\ II\ II$ and $\ell(B_0) = 3$, $\ell(B_1) = 1$, $\ell(B_2) = 3$.

3.4.11 We can see from 3.3.1 and 3.2.22 that $\theta_{n,p} = \psi(p-n,p)(I,II)$. Let a be the unique integer satisfying $a \leq \frac{p}{p-n} < a+1$. Then $\theta_{n,p}$ starts with $I+IIt+IIt^2+\ldots+IIt^{a-1}+It^n$.

3.4.12 Before we give the proof of Lemma 3.4.7 we will prove two lemmas assuming that Lemma 3.4.7 is not true. These lemmas tell us about the length of the substrings B_i and eventually lead to a contradiction.

3.4.13 LEMMA. <u>Let</u> f <u>belong to</u> $\mathcal{A}$ <u>and suppose there exists</u> $x \in S^1$ <u>satisfying condition</u> $B_{n/p}$. <u>Suppose</u> x <u>also satisfies the following conditions</u>.

i) $A(x) = I$.

ii) <u>Neither of the conditions of</u> 3.4.7 <u>are satisfied</u>.

iii) For all i, $f^i(x)$ is not equal to a_f or b_f.

Then for any integer i satisfying $0 \leq i \leq p-n-1$ we have $\ell(B_i) \geq a$.

Proof. Suppose that there exists B_i with $\ell(B_i) < a$. Then without loss of generality we may assume that $\ell(B_0) < a$. Let $B_0 = A_0 \ldots A_{k_1}$. Since by hypothesis $\theta(x) > \downarrow\theta_{n,p}$, we must have $A_j = III$ for some j satisfying $0 < j \leq k_1$. Let J denote the largest integer between 0 and k_1 such that $A_J = III$. Then it is clear that ${}_I\theta(f^J(x)) < \downarrow\theta_{n,p}$, but this gives a contradiction, since the second condition of lemma 3.4.7 is satisfied.

3.4.14 Let $x \in S^1$ satisfy the conditions of 3.4.12. Then it is easy to see the following. Let the itinerary of x be $(A_0, A_1, \ldots, A_k, \ldots)$ and let $A_k = III$. Then A_i cannot equal I for $k \leq i \leq k+a$. Consequently, III cannot belong to B_i if $\ell(B_i) = a$ and if III belongs to B_i and $\ell(B_i) = a+1$ then B_i is the string $I\ III$ followed by $a-1$ II's.

3.4.15 LEMMA. Let f belong to $\mathcal{A}$ and suppose there exists $x \in S^1$ satisfying condition $B_{n/p}$. Suppose x also satisfies the following conditions.

i) $A(x) = I$.

ii) Neither of the conditions of Lemma 3.4.7 are satisfied.

iii) For all i, $f^i(x)$ is not equal to a_f or b_f.

Then for any integer i satisfying $0 \leq i \leq p-n-1$ we have $\ell(B_i) \leqslant a+1$.

Proof. We will suppose for a contradiction that there exists an integer, j, with $\ell(B_j) > a+1$.

Let $p-n;p = a_0 \ldots a_{p-n-1}$. Then recall the following facts from section 3.2. Firstly, $gs(p-n,p) = a_0 \ldots a_{p-n-1}$. Secondly, a_i is equal to either a or $a+1$

Thirdly, $a_0 = a$ and $a_{p-n-1} = a+1$.

Consider $\ell(B_0)\ell(B_1)\ldots\ell(B_{p-n-1})$. This is a(p-n,p)-element as $\sum_0^{p-n-1} \ell(B_i) = p$. Since, gs(p-n,p) equals $a_0 \ldots a_{p-n-1}$ we can find an integer, m , such that $\ell(B_m)\ell(B_{m+1})\ldots\ell(B_{m+p-n-1}) < a_0 \ldots a_{p-n-1}$, where the subscripts are taken modulo $p-n$. Without loss of generality we may assume that $\ell(B_0)\ell(B_1)\ldots\ell(B_{p-n-1}) < a_0 \ldots a_{p-n-1}$. Thus we can find an integer k with $0 \leq k < p-n-1$ such that $\ell(B_0) = a_0$, $\ell(B_1) = a_1 \ldots \ell(B_{k-1}) = a_{k-1}$, $\ell(B_k) < a_k$. The previous lemma shows that $\ell(B_k) = a$ and $a_k = a+1$.

We know by hypothesis that $\theta(x) > {\downarrow}\theta_{n,p}$, so for at least one integer, j , satisfying $0 < j < k$ we have that $\ell(B_j) = a+1$ and that III belongs to B_j . Let J be the largest integer with these properties.

Then we have $\ell(B_J) = a_J, \ell(B_{J+1}) = a_{J+1}, \ldots, \ell(B_{k-1}) = a_{k-1}$, $\ell(B_k) = a_k - 1$. Let $m = 1 + \sum_0^{J-1} a_i$ then it can be seen that ${}_I\theta(f^m(x))$ has the following properties. Firstly $\ell(B_0({}_I\theta(f^m(x))) = a_J - 1$, $\ell(B_1({}_I\theta(f^m(x))) = a_{J+1}$, $\ell(B_2({}_I\theta(f^m(x)) = a_{J+2}$, ..., $\ell(B_{k-J}({}_I\theta(f^m(x))) = a_k - 1$. Secondly, III does not belong to $B_i({}_I\theta(f^m(x)))$ for $0 \leq i \leq k-J$.

We know by hypothesis that ${}_I\theta(f^m(x)) > {\downarrow}\theta_{n,p}$ so we can see that we must have $(a_J-1)a_{J+1}a_{J+2}\ldots(a_k-1) \geq a_0 a_1 \ldots a_{k-J}$. We will now show that $(a_J-1)a_{J+1}\ldots(a_k-1) < a_0 \ldots a_{k-J}$ which gives a contradiction and hence proves the lemma.

From section 3.2 we know that $\ell\ell(p-n,p) = a_{p-n-1}a_1 a_2 \ldots a_{p-n-2}a_0$. Thus $a_J a_{J+1} \ldots a_{J+p-n-1} \leq a_{p-n-1}a_1 a_2 \ldots a_{p-n-2}a_0$, where subscripts are taken modulo $p-n$. Therefore $(a_J-1)a_{J+1}\ldots a_k \leq (a_{p-n-1}-1)a_1 \ldots a_{k-J}$. Using the fact that $a_{p-n-1} - 1 = a_0$ gives

$$(a_J-1)a_{J+1}\ldots(a_k-1) < a_0 a_1 \ldots a_{k-J} \quad .$$

3.4.16 We will now prove Lemma 3.4.7.

Proof. Suppose that $x \in S^1$ satisfies condition $B_{\frac{1}{2}}$, then from 3.0.2 we can see that $\theta(x)$ must equal either $(I+IIIt)(1+t^2)^{-1}$ or $(III-It)(1+t^2)^{-1}$. Without loss of generality we will assume that $\theta(x) = (III-It)(1+t^2)^{-1}$. But $_I\theta(x)$ is then equal to $I+It+(III-It)t^2(1+t^2)^{-1}$ which is less than $+\theta_{1,2}$. So the lemma is true if $p = 2$.

Suppose that we have proved the lemma for all $B_{n/p}$ for which the denominator, p , is less than q . We shall now prove it for $B_{m/q}$. Let x satisfy $B_{m/q}$ and suppose that $f^i(x)$ is not equal to a_f or b_f for any $i \geq 0$; suppose also for a contradiction that neither of the conditions of 3.4.7 are satisfied.

Without loss of generality we may assume that $A(x) = I$. Lemmas 3.4.13 and 3.4.15 tell us that $\ell(B_i)$ is equal to either a or $a+1$, where a is the unique integer given by $a \leq \frac{q}{q-m} < a+1$. We also know that if $\ell(B_i) = a$ then $B_i = I\ II\ II \ldots II$ and if $\ell(B_i) = a+1$ then B_i is either equal to $I\ II\ II \ldots II$ or $I\ III\ II\ II \ldots II$.

We will now consider $B_0 \ldots B_{q-m-1}$ and construct a new string $A_0^1 \ldots A_{q-m-1}^1$ in the following way.

$$A_i^1 = \begin{cases} I & \text{if } \ell(B_i) = a \\ II & \text{if } \ell(B_i) = a+1 \text{ and } B_i = I\ II\ II \ldots II \\ III & \text{if } \ell(B_i) = a+1 \text{ and } B_i = I\ III\ II \ldots II \end{cases} .$$

We can choose $g \in \mathcal{A}$ such that there exists a periodic point $z \in S^1$, of period $q-m$ and with address $(A_0^1, A_1^1, \ldots, A_{q-n-1}^1, A_0^1, \ldots)$. This point, z , will have rotation number $\frac{r}{q-n}$ for some integer r with $0 < r < q-n$.

We have shown how to construct $\theta_g(z)$ from $\theta_f(x)$. Notice that if $A(f^i(x)) = I$, for some $i \geq 0$ and we apply the construction to $\theta_f(f^i(x))$ we obtain $\theta_g(g^j(z))$ for some $j \geq 0$. Notice also that if $A(f^i(x)) = III$ for some $i \geq 0$ and we apply the construction to ${}_I\theta_f(f^i(x))$ we obtain ${}_I\theta_g(g^j(z))$ for some $j \geq 0$.

By the inductive hypothesis we know that z satisfies Lemma 3.4.7. Suppose that the first condition is satisfied i.e. $\theta(g^i(z)) < {+\theta}_{r,q-m}$. Then applying the reverse of the above construction gives $\theta(f^j(x)) < {+\theta}_{m,q}$ for some $j \geq 0$ which gives a contradiction. Similarly if the second condition is satisfied we obtain a contradiction. Thus we have shown that if x satisfies $B_{m/q}$ and $f^i(x)$ is not equal to a_f or b_f for any $i \geq 0$ the conditions of Lemma 3.4.7 are satisfied.

Suppose a satisfies condition $B_{m/q}$ then one of $\theta(a+)$ or $\theta(a-)$ will be periodic and we can apply the above argument. A similar argument applies when b satisfies $B_{m,q}$.

CHAPTER 4 PIECEWISE LINEARIZATION AND THE ROTATION INTERVAL

4.0 INTRODUCTION

In the first section of this chapter we follow the work of Milnor and Thurston [6] to show that given any $f \in \mathcal{A}$ with positive entropy that we can construct another function $F: \frac{[0,1]}{0 \sim 1} \to \frac{[0,1]}{0 \sim 1}$ with the following properties. Firstly, F is piecewise linear, in the obvious sense, and secondly f is topologically semi-conjugate to F.

The function F is determined by two numbers, one of which is the topological entropy of f and the other is defined to be the twist number, denoted $T(f)$. We have shown in the second chapter how to calculate the topological entropy of f from its kneading matrix and in section 4.2 we show how to calculate $T(f)$ from this matrix.

In the third section we show that the rotation intervals of f and F are the same. We then use the results of the third chapter to give a method of calculating this interval.

4.1 CONSTRUCTION OF THE PIECEWISE LINEAR MAP

4.1.0 DEFINITION. Let the set of functions belonging to $\mathcal{A}$ with positive entropy be denoted by $\mathcal{A}^+$.

In this section we will consider a given function, $f \in \mathcal{A}^+$, and show how to construct the map F associated to it.

4.1.1 Recall from 1.3.5 the definitions of $\gamma_b(J)$ and $\gamma_c(J)$. We will denote the sum $\gamma_b(J)+\gamma_c(J)$ by $\gamma(J)$. As in section 2.1 we will let r denote the radius of convergence of $\gamma(S^1)$.

4.1.2 DEFINITION. Given an interval $J \subseteq S^1$. Let

$$\Lambda(J) = \lim_{t\to r} \frac{\gamma(J)}{\gamma(S^1)} \quad .$$

Since $0 \ll \gamma(J) \ll \gamma(S^1)$, (See 2.2.10 for the definition of $\ll$). the limit $\Lambda(J)$ exists and it is clear that $0 \le \Lambda(J) \le 1$.

4.1.3 LEMMA. Let $J \subset S^1$ be an interval with the following properties. Firstly, $f|_J$ is a homeomorphism and secondly, $a \notin f(J)$. Then $\Lambda(f(J)) = \frac{1}{r}\Lambda(J)$.

Proof. Let $\sum_0^\infty \gamma_i(J)t^i$ denote $\gamma(J)$ and $\sum_0^\infty \gamma_i(f(J))t^i$ denote $\gamma(f(J))$. Then for any integer $n \ge 1$ we have $\gamma_n(J)$ is equal to the cardinality of $\{x \in J \cap P_n | f^n(x) = c \text{ or } f^n(x) = b\}$. Since $f|_J$ is a homeomorphism we can see that $\gamma_n(J)$ is equal to the cardinality of $\{x \in f(J) \cap P_{n-1} | f^{n-1}(x) = c \text{ or } f^{n-1}(x) = b\}$, but this is clearly equal to $\gamma_{n-1}(f(J))$.

Thus we have $\sum_1^\infty \gamma_i(J) = t\sum_0^\infty \gamma_i(f(J))$. So in particular $t\gamma(f(J)) \ll \gamma(J) \ll 2+t\gamma(f(J))$.

Therefore $\lim_{t\to r} \frac{t\gamma(f(J))}{\gamma(S^1)} \le \lim_{t\to r} \frac{\gamma(J)}{\gamma(S^1)} \le \lim_{t\to r} \frac{2+t\gamma(f(J))}{\gamma(S^1)}$, which implies that $r\Lambda(f(J)) \le \Lambda(J) \le r\Lambda(f(J))$.

4.1.4 LEMMA. Let J_1 and J_2 be intervals contained in S^1. Suppose that J_1 and J_2 intersect only at a common endpoint. Then $\Lambda(J_1 \cup J_2) = \Lambda(J_1)+\Lambda(J_2)$.

Proof. Let $\sum_0^\infty \gamma_i(J_1)t^i$ denote $\gamma(J_1)$ and $\sum_0^\infty \gamma_i(J_2)t^i$ denote $\gamma(J_2)$. Since $J_1 \cap J_2$ contains only one point we have $0 \le \gamma_n(J_1)+\gamma_n(J_2)-\gamma_n(J_1 \cup J_2) \le 1$. So we must have $0 \ll \gamma(J_1)+\gamma(J_2)-\gamma(J_1 \cup J_2) \ll (1-t)^{-1}$. Thus

$0 \le \lim_{t\to r}[\gamma(J_1)+\gamma(J_2)-\gamma(J_1 \cup J_2)] \le \frac{1}{1-r}$, which gives

$$0 \le \Lambda(J_1)+\Lambda(J_2)-\Lambda(J_1 \cup J_2) \le \frac{1}{1-r} \lim_{t\to r} \frac{1}{\gamma(S^1)} \quad .$$

4.1.2 DEFINITION. Given an interval $J \subseteq S^1$. Let

$$\Lambda(J) = \lim_{t \to r} \frac{\gamma(J)}{\gamma(S^1)} \quad .$$

Since $0 << \gamma(J) << \gamma(S^1)$, (See 2.2.10 for the definition of $<<$). the limit $\Lambda(J)$ exists and it is clear that $0 \leq \Lambda(J) \leq 1$.

4.1.3 LEMMA. <u>Let</u> $J \subset S^1$ <u>be an interval with the following properties</u>. <u>Firstly</u>, $f|_J$ <u>is a homeomorphism and secondly</u>, $a \notin f(J)$. <u>Then</u> $\Lambda(f(J)) = \frac{1}{r}\Lambda(J)$.

<u>Proof</u>. Let $\sum_0^\infty \gamma_i(J)t^i$ denote $\gamma(J)$ and $\sum_0^\infty \gamma_i(f(J))t^i$ denote $\gamma(f(J))$. Then for any integer $n \geq 1$ we have $\gamma_n(J)$ is equal to the cardinality of $\{x \in J \cap P_n | f^n(x) = c \text{ or } f^n(x) = b\}$. Since $f|_J$ is a homeomorphism we can see that $\gamma_n(J)$ is equal to the cardinality of $\{x \in f(J) \cap P_{n-1} | f^{n-1}(x) = c$ or $f^{n-1}(x) = b\}$, but this is clearly equal to $\gamma_{n-1}(f(J))$.

Thus we have $\sum_1^\infty \gamma_i(J) = t\sum_0^\infty \gamma_i(f(J))$. So in particular $t\gamma(f(J)) << \gamma(J) << 2 + t\gamma(f(J))$.

Therefore $\lim_{t \to r} \frac{t\gamma(f(J))}{\gamma(S^1)} \leq \lim_{t \to r} \frac{\gamma(J)}{\gamma(S^1)} \leq \lim_{t \to r} \frac{2 + t\gamma(f(J))}{\gamma(S^1)}$, which implies that $r\Lambda(f(J)) \leq \Lambda(J) \leq r\Lambda(f(J))$.

4.1.4 LEMMA. <u>Let</u> J_1 <u>and</u> J_2 <u>be intervals contained in</u> S^1. <u>Suppose that</u> J_1 <u>and</u> J_2 <u>intersect only at a common endpoint</u>. <u>Then</u> $\Lambda(J_1 \cup J_2) = \Lambda(J_1) + \Lambda(J_2)$.

<u>Proof</u>. Let $\sum_0^\infty \gamma_i(J_1)t^i$ denote $\gamma(J_1)$ and $\sum_0^\infty \gamma_i(J_2)t^i$ denote $\gamma(J_2)$. Since $J_1 \cap J_2$ contains only one point we have $0 \leq \gamma_n(J_1) + \gamma_n(J_2) - \gamma_n(J_1 \cup J_2) \leq 1$. So we must have $0 << \gamma(J_1) + \gamma(J_2) - \gamma(J_1 \cup J_2) << (1-t)^{-1}$. Thus

$0 \leq \lim_{t \to r}[\gamma(J_1) + \gamma(J_2) - \gamma(J_1 \cup J_2)] \leq \frac{1}{1-r}$, which gives

$$0 \leq \Lambda(J_1) + \Lambda(J_2) - \Lambda(J_1 \cup J_2) \leq \frac{1}{1-r} \lim_{t \to r} \frac{1}{\gamma(S^1)} \quad .$$

Since $\lim\limits_{t\to r} \frac{1}{\gamma(S^1)} = 0$, we obtain $\Lambda(J_1)+\Lambda(J_2) = \Lambda(J_1 \cup J_2)$.

4.1.5 Let $x,y \in S^1 \setminus \{a\}$ with $x < y$ (see 1.0) . Then Λ is defined on the interval (x,y) . Using lemma 4.1.4 we can define Λ on the compliment of this interval in the following way. Let $[y,x]$ denote the compliment of (x,y) in S^1 . Then $\Lambda([y,x]) = \Lambda([y,a))+\Lambda([a,x])$. Notice that we have $\Lambda([y,x]) = 1 - \Lambda((x,y))$.

With this observation, the following is an immediate consequence of Lemma 4.1.3.

4.1.6 LEMMA. <u>Let</u> $J \subset S^1$ <u>be an interval or the compliment of an interval such that</u> $f|_J$ <u>is a homeomorphism</u>. <u>Then</u> $\Lambda(f(J)) = \frac{1}{r}\Lambda(J)$.

4.1.7 DEFINITION. Let $\lambda : S^1 \to \frac{[0,1]}{0\sim 1}$ be the map defined by

$$x \to \begin{cases} \Lambda((a,x)) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

4.1.8 LEMMA. λ <u>is continuous</u>.

<u>Proof</u>. Given any $\varepsilon > 0$ we can choose an integer $n \geq 0$ such that $r^n < \varepsilon$. We will show continuity at $x_0 \in S^1$.

Suppose, first, that $f^i(x_0)$ is not equal to a or b for any integer i satisfying $0 \leq i \leq n$. Then we can choose a small neighbourhood (v,w) of x_0 with the same property. Now, $\lambda(x_0) = \Lambda([a,x_0)] = \Lambda((a,v])+\Lambda((v,x_0))$. Therefore, if x belongs to (v,w) we have $|\lambda(x)-\lambda(x_0)| \leq \Lambda((v,w))$. Since, by hypothesis $f^n|_{(v,w)}$ is a homeomorphism lemma 4.1.6 tells us that $\Lambda(v,w)) = r^n\Lambda(f^n(v,w))$ which is less than ε .

If $f^i(x_0)$ is equal to either a or b for some integer i satisfying

$0 \leq i \leq n$, we can choose a small neighbourhood (v,w) of x_0 such that $f^n|_{(v,x_0)}$ is a homeomorphism and $f^n|_{(x_0,w)}$ is a homeomorphism. By choosing an integer, n , such that $r^n < \frac{\varepsilon}{2}$ we can repeat the above argument.

4.1.9 We can easily see that λ is onto and that λ lifts to a monotonic increasing map from $\mathbb{R} \to \mathbb{R}$.

4.1.10 THEOREM. <u>There exists a unique map</u> $F: \frac{[0,1]}{0\sim 1} \to \frac{[0,1]}{0\sim 1}$ <u>with the following properties</u>:

i) $F(\lambda(x)) = \lambda(f(x))$ <u>for any</u> $x \in S^1$.

ii) F <u>is piecewise-linear with slope</u> $\pm \frac{1}{r}$ <u>everywhere</u>.

<u>Proof</u>. We will show first that if $\lambda(x) = \lambda(y)$ then $\lambda(f(x)) = \lambda(f(y))$, in order to show that F is well-defined.

Suppose that $\lambda(x) = \lambda(y)$. Then either $\Lambda(x,y) = 0$ or $\Lambda(y,x) = 0$, we will assume without loss of generality that $\Lambda((x,y)) = 0$. Lemmas 4.1.4 and 4.1.6 then tell us that $\Lambda(f(x,y)) = 0$. Therefore $\lambda(f(x)) = \lambda(f(y))$.

To show existence and uniqueness we will compute F explicitly.

Let x belong to $[a,c)$. Then $F(\lambda(x)) = \lambda(f(x))$. Now $\lambda(f(x)) = \Lambda(f[a,x))+\lambda(f(a))$. By 4.1.3 $\Lambda(f[a,x)) = \frac{1}{r}\Lambda([a,x))$ which by the definition of Λ is equal to $\frac{1}{r}\lambda(x)$. Thus $F(\lambda(x)) = \lambda(f(a)) + \frac{1}{r}\lambda(x)$.

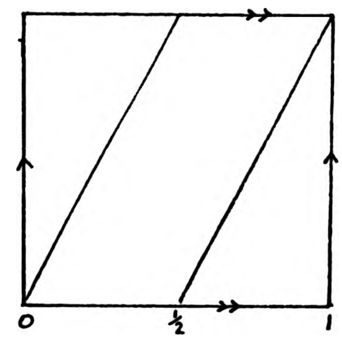
So we can see that $F|_{\lambda[a,c)}$ is linear with slope $+\frac{1}{r}$.

Similar calculations show that if x belongs to $[c,b]$ we have $F(\lambda(x)) = \frac{1}{r}\lambda(x) - \frac{1}{r}\lambda(c)$, and that if x belongs to $[b,a)$ we have $F(\lambda(x)) = \frac{2}{r}\lambda(b) - \frac{1}{r}\lambda(c) - \frac{1}{r}\lambda(x)$. Thus $F|_{\lambda[c,b]}$ is linear with slope $+\frac{1}{r}$ and $F|_{\lambda[b,1)}$ is linear with slope $-\frac{1}{r}$.

4.1.11 LEMMA. <u>None of</u> $\Lambda(a,c)$, $\Lambda(c,b)$ <u>or</u> $\Lambda(b,a)$ <u>are equal to zero</u>.

<u>Proof</u>. Since f belongs to $\mathcal{A}$ we know that $f(b,a) \subset f(a,c)$ and $f(b,a) \subset f(c,b)$. Therefore if either $\Lambda(a,c)$ or $\Lambda(c,b)$ equals zero so must $\Lambda(b,a)$. It is clear that $F : \frac{[0,1]}{0\sim 1} \to \frac{[0,1]}{0\sim 1}$ must be the identity map, but this has slope 1 which contradicts the fact that $r < 1$.

Suppose that $\Lambda(b,a) = 0$. Then since $r < 1$ it is easily seen that we must have $r = \frac{1}{2}$. and $F(0) = 0$ i.e. the graph of F looks like that following.



Thus the limit as x tends upwards to 1 of $F(x)$ is 1 and the limit as x tends downwards to 0 of $F(x)$ is 0 . Therefore $\lim_{x \uparrow a} \lambda(f(x)) = 1$ and $\lim_{x \downarrow a} \lambda(f(x)) = 0$, but this contradicts the fact that a is a minimum of f .

4.1.12 LEMMA. $\lambda(b) = \frac{r+1}{2}$.

<u>Proof</u>. From the proof of theorem 4.1.10 we know that if x belongs to (b,a) we have $\lambda(f(x)) = \frac{2}{r}\lambda(b) - \frac{1}{r}\lambda(c) - \frac{1}{r}\lambda(x)$. Taking the limit as x tends up to a gives the following.

$$\lambda(f(a)) = \frac{2}{r}\lambda(b) - \frac{1}{r}\lambda(c) - \frac{1}{r} \quad . \quad \text{Thus}$$

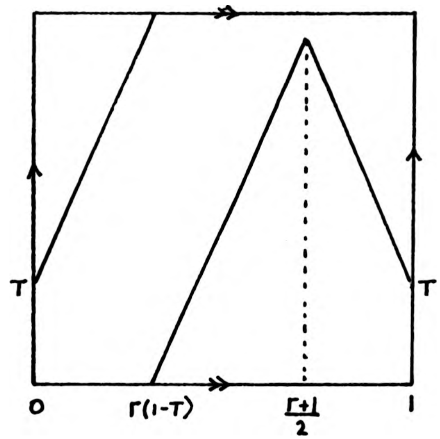
$$\lambda(b) = \frac{r}{2}(\lambda(f(a)) + \frac{1}{r}\lambda(c) + \frac{1}{r}) \quad .$$

Now, $\frac{1}{r}\lambda(c) = \frac{1}{r}\Lambda(a,c)$ which by 4.1.6 can be seen to equal $1-\Lambda(f(c),f(a))$ which equals $1-\lambda(f(a))$. Thus $\lambda(b) = \frac{r}{2}(1+\frac{1}{r})$.

4.1.13 Notice that F is completely determined by r and by $\lambda(f(a))$.

4.1.14 DEFINITION. The twist number of f denoted $T(f)$ is $\lambda(f(a))$.

4.1.15 It is easily seen that F can be thought of as an element of $\mathcal{A}$ and the graph of F looks like the following.



4.2 THE TWIST NUMBER

4.2.0 In this section we show how to calculate the twist number from the entries in the kneading matrix.

We will consider a fixed function f belonging to $\mathcal{A}^+$.

Let $\begin{vmatrix} \nu^{I}(c) & \nu^{I}(b) \\ \nu^{II}(c) & \nu^{II}(b) \end{vmatrix}$ denote the kneading matrix of f and let D denote the kneading determinant.

4.2.1 LEMMA. $t[\gamma_c(S^1) + \gamma_b(S^1)] = \frac{1}{D}[(\nu^{II}(c) - \nu^{II}(b))+(\nu^I(c)-\nu^I(b))-D]$.

<u>Proof</u>. We can see from 1.3.10 that

$$\begin{vmatrix} \nu^I(c) & \nu^I(b) \\ \nu^{II}(c) & \nu^{II}(b) \end{vmatrix} \begin{vmatrix} \gamma_c(S^1) \\ \gamma_b(S^1) \end{vmatrix} = \begin{vmatrix} -\nu^I(a) \\ -\nu^{II}(a) \end{vmatrix} .$$

Therefore
$$\begin{vmatrix} \gamma_c(S^1) \\ \gamma_b(S^1) \end{vmatrix} = \frac{1}{D} \begin{vmatrix} \nu^{II}(b) & -\nu^I(b) \\ -\nu^{II}(c) & \nu^I(c) \end{vmatrix} \begin{vmatrix} -\nu^I(a) \\ -\nu^{II}(a) \end{vmatrix} .$$

Lemma 1.2.4 states that $t\nu(a) = \nu(c)-II+I$, so it is clear that

$$\begin{vmatrix} t\gamma_c(S^1) \\ t\gamma_b(S^1) \end{vmatrix} = \frac{1}{D} \begin{vmatrix} \nu^{II}(b) & -\nu^I(b) \\ -\nu^{II}(c) & \nu^I(c) \end{vmatrix} \begin{vmatrix} -\nu^I(c) - 1 \\ -\nu^{II}(c) + 1 \end{vmatrix} .$$

Multiplying out gives the required result.

4.2.2 LEMMA. $t[\gamma_c((a,c))+\gamma_b((a,c))] = \frac{(1+t)}{2D}[(\nu^I(c)-\nu^I(b))+(1-t)(\nu^{II}(c)-\nu^{II}(b))-D]$.

<u>Proof</u>. From section 1.3 we know that

$$\gamma_c((a,c))\nu(c) + \gamma_b((a,c))\nu(b) = \theta(c-) - \theta(a+) .$$

Which can be written as:

$$\begin{vmatrix} \nu^I(c) & \nu^I(b) \\ \nu^{II}(c) & \nu^{II}(b) \end{vmatrix} \begin{vmatrix} \gamma_c((a,c)) \\ \gamma_b((a,c)) \end{vmatrix} = \begin{vmatrix} \theta_I(c-) - \theta_I(a+) \\ \theta_{II}(c-) - \theta_{II}(a+) \end{vmatrix} .$$

From Lemma 1.2.7 we can see that $t\theta(c-) = t(\frac{I+II}{2} + t\,\frac{I+III}{2} - \frac{\nu(c)}{2})$.

Similarly from Lemmas 1.2.4 and 1.2.5 we obtain

$t\theta(a+) = t(\frac{III+I}{2}) + \frac{I-II+\nu(c)}{2}$. Thus

$$\begin{vmatrix} \nu^{I}(c) & \nu^{I}(b) \\ \nu^{II}(c) & \nu^{II}(b) \end{vmatrix} \begin{vmatrix} t\gamma_c((a,c)) \\ t\gamma_b((a,c)) \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(t+t^2-t\nu^{I}(c))-\frac{1}{2}(1+t+\nu^{I}(c)) \\ \frac{1}{2}(t-t\nu^{II}(c))-\frac{1}{2}(-1+\nu^{II}(c) \end{vmatrix} .$$

Therefore
$$\begin{vmatrix} t\gamma_c((a,c)) \\ t\gamma_b((a,c)) \end{vmatrix} = \frac{1}{D} \begin{vmatrix} \nu^{II}(b) & -\nu^{I}(b) \\ -\nu^{II}(c) & \nu^{I}(c) \end{vmatrix} \begin{vmatrix} \frac{1}{2}[t^2-1-(1+t)\nu^{I}(c)] \\ \frac{1}{2}[(1+t)-(1+t)\nu^{II}(c)] \end{vmatrix} ,$$

and multiplying out gives the required result.

4.2.3 LEMMA. $\lambda(c) = \frac{1+r}{2}\left[1-\lim_{t\to r} \frac{t(\nu^{II}(c) - \nu^{II}(b))}{(\nu^{I}(c)-\nu^{I}(b))+(\nu^{II}(c)-\nu^{II}(b))-D}\right]$.

<u>Proof</u>. By definition $\lambda(c) = \lim_{t\to r} \frac{\gamma_b((a,c))+\gamma_c((a,c))}{\gamma_b(S^1) + \gamma_c(S^1)}$.

Using the results of the previous two lemmas gives

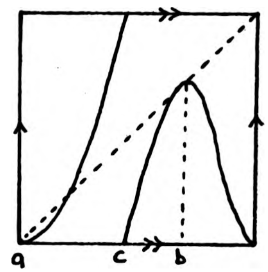
$$\lambda(c) = \lim_{t\to r} \frac{(1+t)[(\nu^{I}(c)-\nu^{I}(b))+(1-t)(\nu^{II}(c)-\nu^{II}(b))-D]}{2[(\nu^{II}(c)-\nu^{II}(b))+(\nu^{I}(c)-\nu^{I}(b))-D]}$$

which reduces to $\lim_{t\to r} [\frac{(1+t)}{2} + \frac{(1+t)(-t(\nu^{II}(c)-\nu^{II}(b))}{2[(\nu^{I}(c)-\nu^{I}(b))+(\nu^{II}(c)-\nu^{II}(b))-D]}$.

4.2.4 We are now in a position to calculate the twist number, T . We know from the previous section that $T = 1 - \frac{\lambda(c)}{r}$. So a simple calculation yields

$$T = \frac{r-1}{2r} + \lim_{t\to r} \frac{(1+r)(\nu^{II}(c)-\nu^{II}(b))}{2[\nu^{I}(c)-\nu^{I}(b)+\nu^{II}(c)-\nu^{II}(b)-D]} .$$

4.2.5 EXAMPLE. Let f belong to $\mathcal{A}$. Suppose that both a and b are fixed points. The graph of f might look like the following.



Easy calculation shows $\theta(c+) = II + It(1-t)^{-1}$,

$\theta(c-) = I + IIIt - It^{2}(1-t)^{-1}$, $\theta(b-) = II(1-t)^{-1}$ and $\theta(b+) = III - II(1-t)^{-1}$.

Thus we have $\nu(b) = (III-II) - 2IIt(1-t)^{-1}$ and $\nu(c) = (II-I)+(I-III)t+I2t^{2}(1-t)^{-1}$

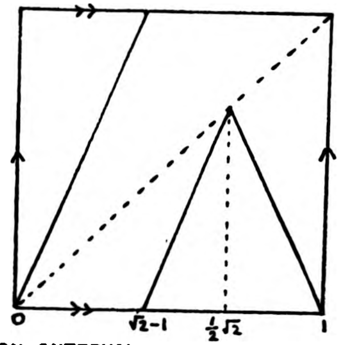
$$\text{So} \quad \begin{vmatrix} \nu^{I}(c) & \nu^{I}(b) \\ \nu^{II}(c) & \nu^{II}(b) \end{vmatrix} = \begin{vmatrix} -1+t+2t(1-t)^{-1} & 0 \\ 1 & -1-2t(1-t)^{-1} \end{vmatrix} .$$

The kneading determinant of f is $\frac{(1+t)}{(t-1)}(t^{2}+2t-1)$. The smallest positive zero occurs when $t = \sqrt{2}-1$, so by 2.1.13 the topological entropy of f is $\log(\sqrt{2}+1)$.

Using the formula given in 4.2.4 to calculate the twist number we obtain

$$T = \frac{\sqrt{2}-2}{-2\sqrt{2}-2} + \lim_{t\to r} \frac{\sqrt{2}(1-\nu^{II}(b))}{2(1-\nu^{II}(b))} ,$$ which is equal to 0 .

Thus the graph of F looks like the following.



4.3 THE ROTATION INTERVAL

4.3.0 Let f belong to $\mathcal{A}^+$ and let F be the piecewise linear *map* associated to f. In this section we will show that the rotation interval of f is equal to the rotation interval of F. Then we give a method for calculating this interval.

4.3.1 Recall from the introduction of this thesis the following. The rotation interval, R_f, of f is the closure of $\{\rho(f,x) \mid x \in S^1$ such that x is periodic$\}$, and $\exp: [0,1) \to S^1$ is the map defined by $\exp(t) = e^{2\pi it}$. For convenience we will write $\bar{x}$ for $\exp^{-1}(x)$.

4.3.2 LEMMA. <u>Suppose that</u> $x \in S^1$ <u>is periodic with respect to</u> f <u>with</u> $\rho(f,x) = \frac{m}{n}$. <u>Then</u> $\rho(f,x) = \rho(F,\lambda(x))$.

<u>Proof</u>. Choose lifts $\bar{f}$ of f and $\bar{F}$ of F such that $0 \le \bar{f}(0) < 1$ and $0 \le \bar{F}(0) < 1$. Then if x belongs to $[a_f, c_f]$ we have $\lambda(x) \in [a_F, c_F]$ so $0 \le \bar{f}(\bar{x}) \le 1$ implies that $0 \le \bar{F}(\overline{\lambda(x)}) \le 1$. Similarly $1 \le \bar{f}(\bar{x}) < 2$ implies that $1 \le \bar{F}(\overline{\lambda(x)}) < 2$. We know that for any $k > 0$ we have $km \le \bar{f}^{kn}(\bar{x}) \le km+1$, simple induction then shows that $km \le \bar{F}^{kn}(\overline{\lambda(x)}) \le km+1$. Thus $km-1 \le \bar{F}^{kn}(\overline{\lambda(x)}) - \lambda(\bar{x}) \le km+1$ and so $\lim_{k\to\infty} \frac{1}{kn}|\bar{F}^{kn}(\overline{\lambda(x)})-\lambda(\bar{x})| = \frac{m}{n}$.
Therefore $\rho(f,x) = \rho(F,\lambda(x)) = \frac{m}{n}$.

4.3.3 COROLLARY. <u>The rotation interval of</u> f <u>is contained in the rotation interval of</u> F .

Thus we can see that if R_F contains a single point then we have $R_F = R_f$. If R_F contains more than a single point we have R_F equals the closure of $\{\rho(F,y) \mid y \in \frac{[0,1]}{0\sim 1}$ such that y is periodic and such that

$F^i(y)$ is not equal to 0 for any $i \geq 0\}$.

4.3.4 LEMMA. <u>Let</u> x <u>belong to</u> S^1 . <u>Suppose that</u> $\lambda(x)$ <u>is periodic with respect to</u> F <u>with</u> $\rho(F,\lambda(x)) = \frac{m}{n}$. <u>Suppose also that</u> $F^i(\lambda(x)) \neq 0$ <u>for any</u> $i \geq 0$. <u>Then</u> $\rho(f,x) = \rho(F,\lambda(x))$.

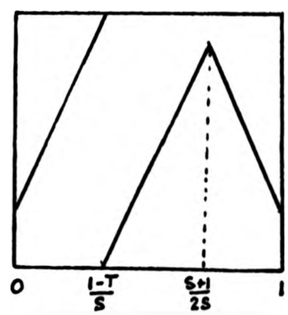
<u>Proof</u>. Choose lifts $\bar{f}$ of f and $\bar{F}$ of F such that $0 \leq \bar{f}(0) < 1$ and $0 \leq \bar{F}(0) < 1$. Since $F^i(\lambda(x)) \neq 0$ for any $i \geq 0$, $\bar{F}^i(\overline{\lambda(x)})$ can never be an integer. The proof now follows the proof of Lemma 4.3.2 once it has been noted that $0 < \bar{F}(\overline{\lambda(x)}) < 1$ implies $0 < f(\bar{x}) < 1$ and $1 < \bar{F}(\overline{\lambda(x)}) < 2$ implies $1 < \bar{f}(x) < 2$.

We have now proved the following.

4.3.5 PROPOSITION. <u>The rotation interval of</u> f <u>is equal to the rotation interval of</u> F .

We now show how to calculate the rotation interval of the piecewise linear map F .

4.3.6 NOTATION. We will let s denote the slope of F . Thus $s = \frac{1}{r}$ and from 4.1.15 we can see that $\lambda(a_f) = 0$, $\lambda(b_f) = \frac{s+1}{2s}$ and $\lambda(c_f) = \frac{1-T}{s}$.



Given a point $x \in (0,1)$ we will let $\theta(x)$ be the invariant coordinate of x with respect to F. We can express $\theta(x)$ in the form $\theta_I I + \theta_{II} II + \theta_{III} III$, (see 1.1.12). Let $\theta_{II}(x,\frac{1}{s})$ denote the real number $\sum_0^\infty \theta_{II}^i (\frac{1}{s})^i$ where $\sum_0^\infty \theta_{II}^i t^i$ is θ_{II} .

4.3.7 LEMMA. <u>Let</u> x <u>be a periodic point of</u> F <u>with period</u> n . <u>Suppose for all integers</u> $i \geq 0$ <u>that</u> $F^i(x)$ <u>belongs to</u> $(0,\frac{s+1}{2s})$. <u>Then</u> $x = s\theta_{II}(x,\frac{1}{s}) - \frac{T}{s-T}$.

<u>Proof</u>. Let $\theta_{II} = [t^{i_1}+t^{i_2}+\ldots+t^{i_k}]\,(1-t^n)^{-1}$.

$$\text{We know that } F(x) = \begin{cases} xs+T & \text{if } x \text{ belongs to } (0,\frac{1-T}{s}) \\ xs+T-1 & \text{if } x \text{ belongs to } (\frac{1-T}{s}, \frac{s+1}{2s}) \end{cases} .$$

Iterating x n-times gives

$$F^n(x) = xs^n + T(s^{n-1}+s^{n-2}+\ldots+1) - s^{n-i_1-1} - s^{n-i_2-1} \ldots\ldots s^{n-i_k-1} .$$

Since $F^n(x) = x$ we have

$$x = \frac{-T(s^{n-1}+\ldots+1)}{s^n-1} + \frac{s^{n-i_1-1}+\ldots+s^{n-i_k-1}}{s^n-1} ,$$

but $\theta_{II}(x,\frac{1}{s}) = \dfrac{(\frac{1}{s})^{i_1}+\ldots+(\frac{1}{s})^{i_k}}{1-(\frac{1}{s})^n}$ which is equal to $\dfrac{s^{n-i_1}+\ldots+s^{n-i_k}}{s^n-1}$

so $x = s\theta_{II}(x,\frac{1}{s}) - \frac{T}{s-T}$.

A similar proof to the one above, using Lemma 3.3.5, gives the following two lemmas.

4.3.8 LEMMA. *Suppose that* 0 *is a periodic point of* F. *Suppose that* $f^i(0)$ *belongs to* $[0,\frac{s+1}{2s}]$ *for every integer* $i \geq 0$. *Then* $0 = s\theta_{II}(0+,\frac{1}{s}) - \frac{T}{s-T}$.

4.3.9 LEMMA. *Suppose that* $\frac{s+1}{2s}$ *is a periodic point of* F. *Suppose that* $f^i(\frac{s+1}{2s})$ *belongs to* $[0,\frac{s+1}{2s}]$ *for every integer* $i \geq 0$. *Then* $\frac{s+1}{2s} = s\theta_{II}(\frac{s+1}{2s}-, \frac{1}{s}) - \frac{T}{s-T}$.

4.3.10 Suppose that x satisfies the conditions of Lemma 4.3.7, with $\theta_{II} = [t^{i_1}+..+t^{i_k}](1-t^n)^{-1}$. We will let P_x denote the polynomial $t^{n-i_1-1}+t^{n-i_2-1}+...+t^{n-i_k-1}$. The following are easily seen to be true.

1) All the coefficients of P_x are equal to $+1$.

2) P_x has degree less than or equal to $n-1$.

3) If x belongs to $(0,\frac{1-T}{s})$ then the degree of P_x is less than $n-1$. If x belongs to $(\frac{1-T}{s},\frac{s+1}{2s})$ then P_x has degree $n-1$.

4) $$P_{F(x)} = \begin{cases} tP_x & \text{if } x \text{ belongs to } (0,\frac{1-T}{s}) \\ tP_x-t^n+1 & \text{if } x \text{ belongs to } (\frac{1-T}{s},\frac{s+1}{2s}) \end{cases}.$$

Thus we can see that if $P_x = \sum_{i=0}^{n-1} a_i t^i$, where a_i is equal to either 0 or 1, then $P_{F(x)} = \sum_{i=0}^{n-1} a_i t^{\sigma(i)}$ and $P_{F^k(x)} = \sum_{i=0}^{n-1} a_i t^{\sigma^k(i)}$, where σ is the cyclic permutation $(0\ 1\ 2\ 3 \ldots n-1)$.

4.2.11 EXAMPLE. Let x be a periodic point with period 5 and invariant coordinate $(I+IIt+IIt^2+It^3+It^4)(1-t^5)$. Then $\theta_{II} = t+t^2$ and so $P_x = t^3+t^2$. We can see that $P_{F(x)} = t^4+t^3$, $P_{F^2(x)} = 1+t^4$, $P_{F^3(x)} = t+1$, $P_{F^4(x)} = t^2+t$

and $P_{F^5(x)} = t^3+t^2$.

4.3.12 LEMMA. Let $P = \sum_{i=0}^{n-1} b_i t^i$ where for any i we have b_i is either equal to 0 or 1 . Suppose that for all k we have $\frac{T}{s-T} < \frac{\Sigma b_i s^{\sigma^k(i)}}{s^n-1} < \frac{s+1}{2s} + \frac{T}{s-T}$. Then there exists a point $x \in (0,\frac{s+1}{2s})$ with period n and with $P_x = P$.

Proof. Let $x_k = \frac{\Sigma b_i s^{\sigma^k(i)}}{s^n-1} - \frac{T}{s-T}$. It is clear that, for any k , x_k belongs to $(0,\frac{s+1}{2s})$. We will show that $x_k = F^k(x_0)$. To prove this it is sufficient to show the following.

1) If $\Sigma b_i s^{\sigma^k(i)}$ has degree less than $n-1$ then x_k belongs to $(0,\frac{1-T}{s})$.

2) If $\Sigma b_i s^{\sigma^k(i)}$ has degree $n-1$ then x_k belongs to $(\frac{1-T}{s}, \frac{s+1}{2s})$.

Suppose that $\Sigma b_i s^{\sigma^k(i)}$ has degree less than $n-1$. Then $\Sigma b_i s^{\sigma^{(k+1)}(i)} = s\ \Sigma b_i s^{\sigma^k(i)}$. Since $\frac{\Sigma b_i s^{\sigma^{(k+1)}(i)}}{s^n-1} < \frac{s+1}{2s} + \frac{T}{s-T}$ we have $\frac{s\Sigma b_i s^{\sigma^k(i)}}{s^n-1} < \frac{s+1}{2s} + \frac{T}{s-T}$. Which implies that $sx_k < \frac{s+1}{2s} - T$, and since $\frac{s+1}{2s}$ is less than 1 we must have $x_k \in (0,\frac{1-T}{s})$. So we have proved the first statement.

Suppose that $\Sigma b_i s^{\sigma^k(i)}$ has degree $n-1$. Then $\Sigma b_i s^{\sigma^{k+1}(i)} = s\Sigma b_i s^{\sigma^k(i)} - s^n+1$. We know that $\frac{\Sigma b_i s^{\sigma^{k+1}(i)}}{s^n-1} > \frac{T}{s-T}$ which implies that

$$\frac{s\Sigma b_i s^{\sigma^k(i)} - s^n+1}{s^n-1} > \frac{T}{s-T}$$

. This inequality reduces to $x_k > \frac{1-T}{s-T}$, which proves the second statement and hence completes the proof.

4.3.13 A similar proof to the one above gives the following two lemmas.

LEMMA. *Let* $P = \sum_{i=0}^{n-1} b_i t^i$ *where for any* i *we have* b_i *is either equal to* 0 *or* 1. *Suppose that* $\frac{\sum_0^{n-1} b_i s^i}{s^n - 1} = \frac{T}{s-1}$ *and that for any* k *we have*
$\frac{T}{s-1} \leq \frac{\sum_0^{n-1} b_i s^{\sigma^k(i)}}{s^n-1} \leq \frac{s+1}{2s} + \frac{T}{s-1}$. *Then* 0 *is periodic and* $P_{0+} = P$.

LEMMA. *Let* $P = \sum_{i=0}^{n-1} b_i t^i$ *where for any* i *we have* b_i *is either equal to* 0 *or* 1. *Suppose that* $\frac{\sum_0^{n-1} b_i s^i}{s^n - 1} = \frac{s+1}{2s} + \frac{T}{s-1}$ *and that for any* k *we have*
$\frac{T}{s-1} \leq \frac{\sum_0^{n-1} b_i s^{\sigma^k(i)}}{s^n-1} \leq \frac{s+1}{2s} + \frac{T}{s-1}$. *Then* $\frac{s+1}{2s}$ *is periodic and* $P_{\frac{s+1}{2s}-} = P$.

4.3.14 DEFINITION. Let m and n be two positive coprime integers with $m < n$. Let $m;n = a_0 a_1 \ldots a_{m-1}$. Then we define $\psi_t(m,n)$ to be the polynomial $1+t^{i_1}+t^{i_2}+\ldots+t^{i_{m-1}}$ where $i_j = \sum_{r=0}^{j-1} a_r$.

4.3.15 EXAMPLE. $3;5 = 122$. Thus $\psi_t(3,5) = 1+t+t^3$.

4.3.16 LEMMA. *Let* $x \in S^1$ *be periodic with* $\theta(x) = \theta_{m,n}$. *Then* $P_x = \psi_t(m,n)$.

Proof. Let $m;n = a_0 \ldots a_{m-1}$. From 3.4.11 we know that $\theta_{m,n} = \psi(n-m;n)(I,II)$. Lemma 3.2.30 shows that $\psi(n-m;n)(I,II) = [\psi(m;n)(II,I)]^{-1}$. Let θ denote $\psi(m,n)(II,I)$. Then $\theta_{II} = 1+t^{i_1}+t^{i_2}+..+t^{i_{m-1}}$ where $i_j = \sum_{r=0}^{j-1} a_r$. Therefore $\theta_{II}(x) = t^{n-1}+t^{n-1-i_1}+t^{n-1-i_2}+\ldots+t^{n-1-i_{m-1}}$ and so we obtain $P_x = \psi_t(m,n)$.

4.3.17 NOTATION. We will let $\psi_s(m,n)$ denote the real number $\sum_{i=0}^{n-1} a_i s^i$, where $\sum_{i=0}^{n-1} a_i t^i = \psi_t(m,n)$.

4.3.18 THEOREM. $\frac{m}{n}$ <u>belongs to the rotation interval of F if and only if</u>

$$\frac{1}{s-1} \leq \frac{\psi_s(m,n)}{s^n-1} \leq \frac{s+1}{2s} + \frac{1}{s-1} - \frac{s^{n-1}-1}{s^n-1} .$$

<u>Proof</u>. Suppose that $\frac{1}{s-1} \leq \frac{\psi_s(m,n)}{s^n-1} \leq \frac{s+1}{2s} + \frac{1}{s-1} - \frac{s^{n-1}-1}{s^n-1}$.

Let $\psi_t(m,n) = \sum_0^{n-1} b_i t^i$. Then by Lemma 3.2.27 and using the fact that $\ell\ell(n,p) = (n;p)^{-1}$ we obtain $|\Sigma b_i s^{\sigma^k(i)} - \Sigma b_i s^i| \leq s^{n-1}-1$. Therefore one of Lemmas 4.3.12 or 4.3.13 must be satisfied and so there exists a periodic point with rotation number $\frac{m}{n}$.

The reverse implication is an immediate consequence of theorem 3.4.0 and lemmas 4.37, 4.3.8 and 4.3.9.

4.3.19 EXAMPLE. Let f belong to A . Suppose that $T(f) = 0$ and the entropy of f is $\log 2$. We will show that $[0,\frac{5}{11}]$ is contained in the rotation interval and that $\frac{6}{11}$ is not.

$\frac{m}{n}$	$m;n$	$\psi_t(m,n)$	$\psi_2(m,n)$
$\frac{1}{11}$	11	1	1
$\frac{2}{11}$	56	$1+t^5$	33
$\frac{3}{11}$	344	$1+t^3+t^7$	137
$\frac{4}{11}$	2333	$1+t^2+t^5+t^8$	293
$\frac{5}{11}$ -	22223	$1+t^2+t^4+t^6+t^8$	337
$\frac{6}{11}$	122222	$1+t+t^3+t^5+t^7+t^9$	713

Theorem 4.3.18 tells us that $\frac{m}{n}$ belongs to R_f if

$0 \leq \frac{\psi_2(m,n)}{2047} \leq \frac{3}{4} - \frac{1023}{2048}$. Inspection then shows that $\frac{5}{11} \in R_f$ and $\frac{6}{11} \notin R_f$.

CHAPTER 5 PERIODIC ORBITS

5.0 INTRODUCTION

In this chapter we shall be concerned with the existence of periodic points.

Let $\mathbb{N}$ denote the set of all positive integers. We define the $1\times$ Sarkovskii ordering, $\lhd_1$, on $\mathbb{N}$ as follows. If k and ℓ are odd integers greater than 2 then $2^n k \lhd_1 2^m \ell$ if $n > m$ or $n = m$ and $k > \ell$. If $k > 2$ is odd then $2^n \lhd_1 2^n k$. Finally, $2^m \lhd_1 2^n$ if $m < n$. Thus we have

$$1 \lhd_1 2 \lhd_1 4 \lhd_1 8 \lhd_1 \ldots \lhd_1 4.3 \lhd_1 \ldots \lhd_1 2.7 \lhd_1 2.5 \lhd_1 2.3 \lhd_1 \ldots \lhd_1 9 \lhd_1 7 \lhd_1 5 \lhd_1 3 \ .$$

Let $n \times \mathbb{N}$ denote the set of positive integers divisible by n. We define the nxSarkovskii ordering, $\lhd_n$, on $n \times \mathbb{N}$ by $p \lhd_n q$ if and only if $\frac{p}{n} \lhd_1 \frac{q}{n}$. We extend this ordering to $\mathbb{N}$ in the following way. Let $p \lhd_n q$ if p is divisible by n and q is not. Finally, let $p =_n q$ if both p an- q are not divisble by n. Thus we have

$$2 \lhd_2 4 \lhd_2 \ldots \lhd_2 14 \lhd_2 10 \lhd_2 6 \lhd_2 3 =_2 5 =_2 7 \ldots \ .$$

If f has a periodic point of period k implies that f has a periodic point of period ℓ for every ℓ satisfying $\ell \lhd_n k$, we say that f has $n\times$Sarkovskii's ordering of periodic points.

In 1964 Sarkovskii [9] proved the following.

5.0.1 THEOREM. <u>Any continuous function</u> $f:[0,1] \to [0,1]$ <u>has</u> 1 <u>Sarkovskii's ordering of periodic points</u>.

In this chapter we prove,

5.0.2 THEOREM. Let f belong to $\mathcal{A}$. Then we can find two neighbourhoods, U and V, and two coprime integers, n and p, such that the following are true.

1) If x is a periodic point then there exists a positive integer, i, such that $f^i(x) \in U \cup V$ or x has period p or n.

2) If x belongs to U and $f^{p+kn}(x) = x$ for some $k > 0$, then for any $m > k$ there exists a periodic point with period $p+mn$.

3) We have $n\times$Sarkovskii's ordering of points in V.

4) If x is periodic and there exists positive integers i and j such that $f^i(x) \in U$ and $f^j(x) \in V$, then for any positive integer k there exists a periodic point in V of period nk.

5.0.3 Let $\exp: [0,1) \to S^1$ be the map given by $\exp(t) = e^{2\pi it}$. Given $f: S^1 \to S^1$ we will let $\bar{f}: [0,1) \to [0,1)$ denote the map $\exp^{-1} \circ f \circ \exp$. It is easily seen that f belongs to $\mathcal{A}$ if and only if $\bar{f}$ has the following properties. There exist points c and b with $0 < c < b < 1$ such that

i) $\bar{f}|_{[0,c)}$ is continuous and strictly monotone increasing,

ii) the limit of $\bar{f}(x)$ as x tends upwards to c is 1,

iii) $\bar{f}|_{[c,1)}$ is continuous,

iv) $\bar{f}|_{[c,b]}$ is strictly monotone increasing,

v) $\bar{f}|_{[b,1)}$ is strictly monotone decreasing,

vi) $\bar{f}(c) = 0$ and,

vii) the limit of $\bar{f}(x)$ as x tends up to 1 is $\bar{f}(0)$.

In this chapter it will be convenient to identify $\mathcal{A}$ with the set of functions from $[0,1)$ to itself that satisfy the above conditions.

The proof of Theorem 5.0.2 is divided into four sections. In section 5.1

we prove some basic lemmas that are needed in the later sections. In 5.2 we give a proof of the theorem for the special case when both n and $p = 1$, where n and p are the integers in the statement of 5.0.2. In 5.3 we prove the theorem in another special case, when $n = 2$ and $p = 1$. Finally, in section 5.4 we complete the proof.

5.1 BASIC LEMMAS

5.1.0 In this section we prove two lemmas that are necessary for the later sections.

5.1.1 LEMMA. Let $f:\mathbb{R} \to \mathbb{R}$ be a map that is not necessarily continuous but has the following properties: There exist real numbers a_1, a_2, a_3, a_4 with $0 < a_2 < a_3 \le a_1 < 1$ and $a_4 < 1$ such that

i) $f|_{(a_1,1]}$ is an orientation preserving homeomorphism onto $(0,1]$,

ii) f maps $[a_2,a_3]$ continuously onto $[a_4,1]$,

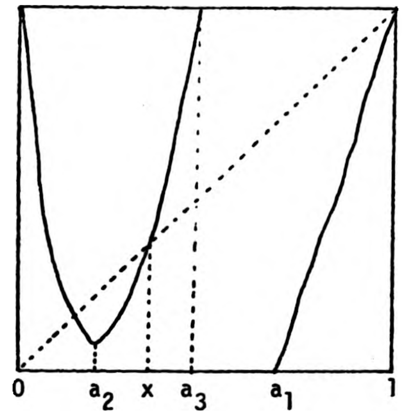
iii) there exists $x \in [a_2,a_3]$ and a positive integer, n, with $f^n(x) \le a_3$.

Then for any positive integer, k, with $k > n$ there exists $x_k \in [a_2,a_3]$ with the properties:

i) x_k is periodic of period k.

ii) If $f^r(x_k) \neq x_k$ then $f^r(x_k)$ is an element of $(a_1,1]$.

5.1.2 EXAMPLE. If the graph of f restricted to $[0,1]$ looks like the following,



then the proposition tells us that f has periodic points of all periods.

5.1.3 PROOF OF LEMMA 5.1.1

$f|_{(a_1,1]}$ is a homeomorphism onto $(0,1]$. Therefore, for any positive integer, i, we can choose an interval $[b_i,c_i]$ contained in $(a_1,1]$ such that $f([b_1,c_1]) = [a_2,a_3]$ and $f([b_i,c_i]) = [b_{i-1},c_{i-1}]$.

As $a_3 \leq a_1$ we have $f(f^{i-1}(c_i)) \leq f(f^{i-1}(b_{i+1}))$. But $f^{i-1}(c_i)$ and $f^{i-1}(b_{i+1})$ are elements of $(a_1,1]$ and since $f|_{(a,1]}$ is orientation preserving we can see that $c_i \leq b_{i+1}$. So, in particular, it is clear that if $[b_i,c_i]$ is contained in $[a_4,1]$ then $[b_{i+k},c_{i+k}] \subset [a_4,1]$, where k is any positive integer.

Next we will show that $[b_n,c_n]=B_n$ is contained in $[a_4,1]$. By the above argument it will be sufficient to show that $c_{n-1} \geq a_4$.

We are told there exists x in $[a_2,a_3]$ with $f^n(x) \leq a_3$. Suppose that $c_{n-1} < f(x)$. Then using the fact that $f|_{(a_1,1]}$ is orientation preserving we can see that $f^{n-1}(c_{n-1}) < f^{n-1}(f(x))$; but this implies that $a_3 < f^n(x)$, which gives a contradiction. So we must have $c_{n-1} \geq f(x)$. As x lies in $[a_2,a_3]$ $f(x)$ must belong to $[a_4,1]$. Therefore $c_{n-1} \geq a_4$.

We have now shown that for any non-negative integer, k, B_{n+k} is contained in $[a_4,1]$. By definition $f^{n+k}(B_{n+k}) = [a_2,a_3]$, so $f^{n+k+1}(B_{n+k}) = [a_4,1]$. Hence we have $B_{n+k} \subset f^{n+k+1}(B_{n+k})$. The fixed point theorem then tells us that there must exist $x \in [a_2,a_3]$ with $f^{n+k+1}(x) = x$. It is clear that if $f^r(x) \neq x$ then $f^r(x)$ is an element of $(a_1,1]$.

5.1.4 Consider $f:[0,1) \to [0,1)$ a map that is not necessarily continuous, but has the property that for some closed interval, $[r,s]$, contained in $[0,1)$, $f([r,s])$ has empty intersection with $[r,s]$.

Then if we define $f_1:[0,1) \to [0,1)$

$$\text{by} \quad f_1(x) = \begin{cases} f(x) & \text{if } x \notin [r,s] \\ f^2(x) & \text{if } x \in [r,s] \end{cases}$$

the following lemma is true.

5.1.5 LEMMA 1) x _is a periodic point of_ f_1 _if and only if_ x _is a periodic point of_ f .

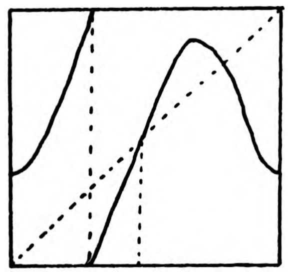
2) _If_ x _is periodic of period_ n _with respect to_ f_1 , _then_ x _has period_ $n+m$ _with respect to_ f ; _where_ m _is the number of elements in_ $\{i | f^i(x) \in [r,s]$ _for_ $1 \leq i < n\}$.

Proof. Suppose that x is periodic of period n with respect to f_1 . Then the cardinality of $\{f^i(x)\}_{i \in N}$ is less than or equal to $2n$ and therefore x is periodic with respect to f . If y is periodic of period n with respect to f , then the cardinality of $\{f_1^i(y)\}_{i \in N}$ is less than or equal to n , and therefore y is periodic with respect to f_1 . Thus, we have proved the first statement.

To prove the second statement, first notice that since $f_1^n(x) = x$ we must have $f^{n+m}(x) = x$. Suppose that $f^p(x) = x$ for some $p < n+m$. Then the following must be true: (1) p divides $n+m$; (2) $p \geq n$, since the orbit of x under f_1 gives n distinct points; (3) $m \leq n$. So $p = n = m$. Therefore for any positive integer, i , we have $f^i(x) \in [r,s]$, but this gives a contradiction, because if $f^i(x) \in [r,s]$ then by the definition of $[r,s]$, $f^{i+1}(x) \notin [r,s]$.

5.2 FIRST SPECIAL CASE OF THEOREM 5.0.2

5.2.1 In this section we will prove Theorem 5.0.2 for the special case when there exists a point, z , in $[c,b]$ with $f(z) = z$. For example the graph of f could look like this:



5.2.2 PROPOSITION. Let f belong to $\mathcal{A}$ and suppose that there exists a point, z , in $[c,b]$ such that $f(z) = z$. Then there exist neighbourhoods U of c and V of b with the following properties:

1) If $f^n(x) = x$ for some $n \geq 0$, then either there exists an integer $i > 0$ with $f^i(x) \in U \cup V$ or x is a fixed point.

2) If $x \in U$ and $f^n(x) = x$, then for any integer k with $k > n$ there exists a point, $y_k \in U$, such that y_k is periodic with period k.

3) We have Sarkovskii's ordering of periodic points in V .

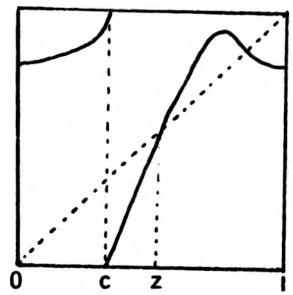
4) <u>If</u> x <u>is periodic and there exists positive integers</u> i <u>and</u> j <u>with</u> $f^i(x) \in U$ <u>and</u> $f^j(x) \in V$, <u>then there exist periodic points of all periods in</u> V.

5.2.3 We will split the proof of this proposition into two cases.

CASE 1. $f(0) \geq v$ for all $v \in [c,b]$ with the property that $f(v) = v$.

CASE 2. There exists a point, v in $[c,b]$ with $f(v) = v$ and $v > f(0)$.

5.2.4 <u>Proof of proposition 5.2.2 when</u> f <u>satisfies the conditions of case</u> 1.



LEMMA A. <u>If</u> $z \in [c,b]$ <u>is such that</u> $f(z) = z$, <u>then</u> f <u>maps</u> $[z,1)$ <u>into</u> $[z,1)$.

<u>Proof</u>. $$f([z,b]) = [f(z),f(b)] = [z,f(b)] \subset [z,1)$$

and $$f([b,1)) = (f(0),f(b)] \subset [z,1) \quad .$$

LEMMA B. <u>If</u> x <u>is a periodic point then either</u> $x \in [z,1)$ <u>or</u> x <u>is a fixed point</u>.

<u>Proof</u>. Suppose that a periodic point x is not contained in $[z,1)$. Then either $x \in [0,c)$ or $x \in [c,z)$.

If $x \in [0,c)$ then $f(x) \in [f(0),1) \subset [z,1)$, but lemma A tells us that

x cannot be periodic. If $x \in [c,z)$ then $f(x) \in [f(c),f(z)) = [0,z)$. We have just shown that $f(x)$ cannot be in $[0,c)$, so we must have $f(x) \in [c,z)$. Similarly for any positive integer, k, we have $f^k(x) \in [c,z)$. Since f restricted to $[c,z]$ is monotone increasing it is clear that $f(x) = x$.

Now we use Lemmas A and B to complete the proof. Lemma B tells us that we only need consider f restricted to $[z,1)$. But since $f|_{[z,1)}$ is continuous and f maps $[z,1)$ into $[z,1)$, we have the Sarkovskii ordering of periodic points in $[z,1)$. The only other periodic points are fixed points in $[c,z]$.

So if we take $V = [z,1)$ and U a small neighbourhood of c containing no fixed points, we are finished.

5.2.5 Proof of Proposition 5.2.2 when f satisfies the conditions of Case 2.

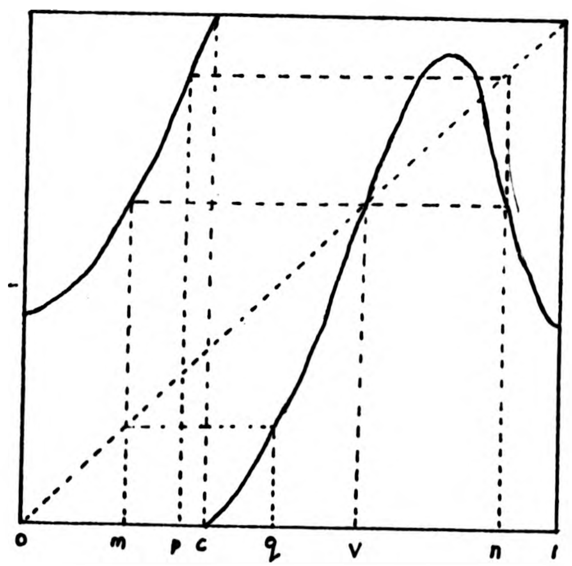
From the facts that $f(0) < v$ and $f(b) > v$, it is easily seen that we can construct the points m,n,p,q satisfying the following conditions:

1) $m \in [0,c)$ and $f(m) = v$,

2) $n \in [b,1)$ and $f(n) = v$,

3) $p \in [0,c)$ and $f(p) = n$,

4) $q \in [c,v]$ and $f(q) = m$.



LEMMA A. <u>If</u> x <u>is a periodic point then one of the following conditions must be satisfied</u>:

1) $f(x) = x$,

2) <u>there exists a positive integer</u>, i , <u>such that</u> $f^i(x) \in [p,q]$,

3) <u>there exists a positive integer</u>, i , <u>such that</u> $f^i(x) \in [v,n]$.

<u>Proof</u>. Suppose that x is a periodic point not satisfying any of the conditions above. Then $x \in [0,m] \cup [m,p) \cup (q,v) \cup (n,1)$.

If $x \in [m,p)$ then $f(x) \in [v,n]$, which gives a contradiction, as $f(x)$ satisfies condition 3.

If $x \in (q,v)$ then $f(x) \in (m,v)$. We have just shown that $f(x)$ cannot lie in $[m,p)$ and by hypothesis $f(x)$ cannot lie in $[p,q]$, so $f(x)$ is an element of (q,v) . Similarly for any positive integer, i , we have that $f^i(x)$ is an element of (q,v) . Since f restricted to (q,v) is monotonic it is clear that $f(x) = x$. Thus we have a contradiction as condition (1) is satisfied.

If $x \in [0,m]$ then $f(x) \in [0,v]$. We know that $f(x)$ cannot be an element of $[m,p)$, $[p,q]$ or (q,v) so $f(x)$ is an element of $[0,m]$. Similarly for any positive integer, k , $f^k(x) \in [0,m]$. Since f restricted to $[0,m]$ is monotonic, we have $f(x) = x$, which again gives a contradiction.

If $x \in (n,1)$ then $f(x) \in [0,v]$, but the argument above gives a contradiction.

Let V denote $[v,n]$ and U denote $[p,q]$. If $f(b) \in V$, we have Sarkovskii's ordering of periodic points in V . If $f(b) \notin V$ then it can be seen from Lemma 5.1.1 that there exist periodic points of all periods in V .

So to prove the proposition we just need to prove the following: <u>Let</u> $y \in [p,q]$ <u>be a periodic point of period</u> n . <u>Then for any integer</u>, k , <u>with</u> k <u>greater than</u> n , <u>there exists a periodic point</u> $y_k \in [p,q]$ <u>of period</u> k .

Define the map $f_1:[0,1) \to [0,1)$ by $f_1(x) = \begin{cases} f(x) & \text{if } x \notin [p,q] \\ f^2(x) & \text{if } x \in [p,q] \end{cases}$.

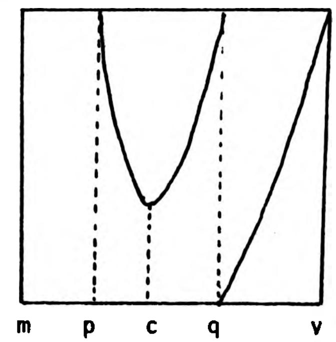
The following are easily shown to be true:

1) $f_1|_{(q,v]}$ is an orientation preserving homeomorphism onto $(m,v]$.

2) f_1 maps $[p,q]$ continuously onto $[f(0),v]$.

3) $m < p < q < v$ and $f(0) < v$. (Note that $f(0)$ is not necessarily greater than m).

So f_1 restricted to $(m,v]$ satisfies the conditions of Lemma 5.1.1. The graph of $f_1|_{(m,v]}$ might look like



If $f_1(c) \leq c$ then by Lemma 5.1.1 f_1 has periodic points of all periods and Lemma 5.1.5 tells us that f has periodic points of all periods greater than 2.

Suppose that $f_1(c) > c$ and that there exists a periodic point $y \in [p,q]$ of period n with respect to f . Then Lemma 5.1.5 tells us that y is a periodic point of period less than or equal to $n-1$ with respect to f_1 . So, by Lemma 5.1.1, f_1 has periodic points of all periods greater than $n-1$ and Lemma 5.1.5 then shows that f has periodic points of all periods greater than n.

5.2.6 In the proof of the second case of Proposition 5.2.5 it should be noticed that we never used the fact that $f(b)$ was less than 1 . So the proof can be repeated to prove:

5.2.7 PROPOSITION. Let $f : \mathbb{R} \to \mathbb{R}$ be a map with the following properties:

1) $f|_{[0,1)}$ satisfies all the properties for being an element of $\mathcal{A}$ except that $f(b) \geq 1$.

2) There exists $v \in [c,b]$ with $f(v) = v$ and $v > f(0)$.

Then there exist neighbourhoods U of c and V of b such that the following are true.

1) If $x \in [0,1)$ and $f^n(x) = x$ then either there exists a positive integer, k , with $f^k(x) \in U \cup V$, or x is a fixed point.

2) If $x \in U$ and $f^n(x) = x$, then for any integer, k , with $k > n$ there exists a point, $y_k \in U$, such that y_k is a periodic point of period k .

3) We have periodic points of all periods in V .

5.3 SECOND SPECIAL CASE OF THE THEOREM

In this section we prove another special case of the theorem, namely:-

5.3.1 PROPOSITION. Let $f \in \mathcal{A}$ satisfy the following properties:

1) $z \in [c,b]$ implies that $f(z) \neq z$; and

2) $f(0) < c < f(b) < b$.

Then there exist neighbourhoods U and V such that the following are true.

i) If $f^n(x) = x$ then either there exists a positive integer, i , with $f^i(x) \in U \cup V$, or x has period 1 or 2.

ii) If $x \in U$ and $f^n(x) = x$, where n is a positive odd integer, then for any odd integer, k , with $k > n$ there exists a point

$y_k \in U$ <u>such that</u> y_k <u>is periodic of period</u> k .

iii) <u>We have</u> 2×<u>Sarkovskii's ordering of periodic points in</u> V .

iv) <u>If</u> x <u>is periodic and there exist positive integers</u> i <u>and</u> j <u>with</u> $f^i(x) \in U$ <u>and</u> $f^j(x) \in V$ <u>then there exist points of all even periods in</u> V .

5.3.2 DEFINITION. Let f belong to $\mathcal{A}$. Then B_f is defined by $B_f := \{z \in [0,c) \mid f^2(z) = z$ and $f(z) \in [c,b]\}$.

5.3.3 LEMMA. <u>Let</u> $f \in \mathcal{A}$ <u>and satisfy conditions</u> (1) <u>and</u> (2) <u>of Proposition 5.3.1. Then</u> B_f <u>is non-empty</u>.

<u>Proof</u>. Since $f(0) < c$ we can find a closed interval, J , contained in $[0,c)$ such that $f(J) = [c,b]$. So $f^2(J) = [f(c),f(b)] = [0,f(b)]$. As $f(b) > c$, $[0,f(b)]$ must contain $[0,c)$. Thus $f^2(J)$ contains $[r,s]$ and we are finished by the fixed point theorem.

5.3.4 We will split the proof of Proposition 5.3.1 into three cases and deal with each one separately.

CASE 1. There exists $z \in B_f$ with $z > f(0)$.

CASE 2. For any $z \in B_f$ we have $z \leq f(0)$, but there exists $z' \in B_f$ with $f(b) > f(z')$.

CASE 3. For any $z \in B_f$ we have $z \leq f(0)$ and $f(z) \geq f(b)$.

5.3.4 <u>Proof of Proposition 5.3.1 when</u> f <u>satisfies the conditions of case 1</u>.

If y is a point in $[c,1)$ then $f(y) < y$. So we can see that for any periodic point x , there must exist a positive integer i for which $f^i(x)$ belongs to $[0,c)$. Thus we can choose the neighbourhoods U and V to be contained in $[0,c)$.

$y_k \in U$ <u>such that</u> y_k <u>is periodic of period</u> k .

iii) <u>We have</u> 2×<u>Sarkovskii's ordering of periodic points in</u> V .

iv) <u>If</u> x <u>is periodic and there exist positive integers</u> i <u>and</u> j <u>with</u> $f^i(x) \in U$ <u>and</u> $f^j(x) \in V$ <u>then there exist points of all even periods in</u> V .

5.3.2 DEFINITION. Let f belong to $\mathcal{A}$. Then B_f is defined by $B_f := \{z \in [0,c) \mid f^2(z) = z$ and $f(z) \in [c,b]\}$.

5.3.3 LEMMA. <u>Let</u> $f \in \mathcal{A}$ <u>and satisfy conditions</u> (1) <u>and</u> (2) <u>of Proposition 5.3.1. Then</u> B_f <u>is non-empty</u>.

<u>Proof</u>. Since $f(0) < c$ we can find a closed interval, J , contained in $[0,c)$ such that $f(J) = [c,b]$. So $f^2(J) = [f(c),f(b)] = [0,f(b)]$. As $f(b) > c$, $[0,f(b)]$ must contain $[0,c)$. Thus $f^2(J)$ contains $[r,s]$ and we are finished by the fixed point theorem.

5.3.4 We will split the proof of Proposition 5.3.1 into three cases and deal with each one separately.

CASE 1. There exists $z \in B_f$ with $z > f(0)$.

CASE 2. For any $z \in B_f$ we have $z \leq f(0)$, but there exists $z' \in B_f$ with $f(b) > f(z')$.

CASE 3. For any $z \in B_f$ we have $z \leq f(0)$ and $f(z) \geq f(b)$.

5.3.4 <u>Proof of Proposition 5.3.1 when</u> f <u>satisfies the conditions of case 1</u>.

If y is a point in $[c,1)$ then $f(y) < y$. So we can see that for any periodic point x , there must exist a positive integer i for which $f^i(x)$ belongs to $[0,c)$. Thus we can choose the neighbourhoods U and V to be contained in $[0,c)$.

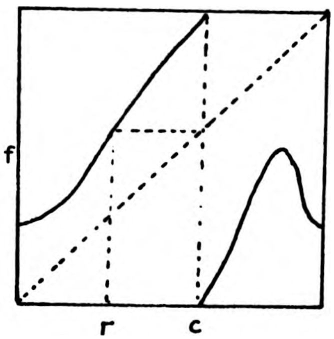
Since $f(0) < c$ we can find a point $r \in [0,c)$ with $f(r) = c$. Clearly if $x \in [r,c)$ then $f(x) \notin [r,c)$ since $f(x)$ is greater than c.

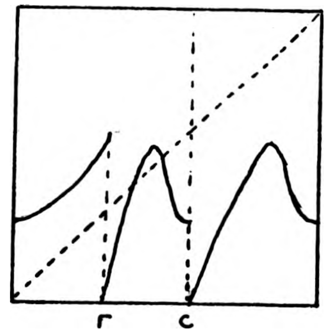
Define $f_1 : [0,1) \to [0,1)$ by

$$f_1(x) = \begin{cases} f(x) & \text{if } x \notin [r,c) \\ f^2(x) & \text{if } x \in [r,c) . \end{cases}$$

e.g.

Graph of f



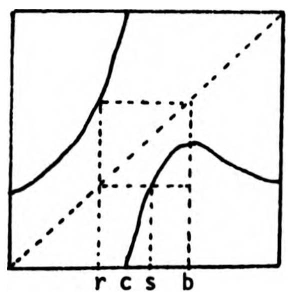


Graph of f_1

It is easily seen that f_1 restricted to $[0,c)$ either satisfies case 2 of Proposition 5.2.2 or satisfies the condition for Proposition 5.2.7. After a moments thought it is seen that Lemma 5.1.5 completes the proof.

5.3.5 <u>Proof of Proposition 5.3.1 when</u> f <u>satisfies the conditions of case 2</u>.

As $f(0) < c$ we can find a point $r \in [0,c)$ with $f(r) = b$. Similarly since $f(b) > c$ we can find a point $s \in [c,b]$ with $f(s) = r$.



Graph of f .

First we will show that we only need to consider the periodic points lying in $(r,b]$.

LEMMA. Let x be a periodic point. Then there exists a positive integer, i , such that $f^i(x) \in (r,b]$.

Proof. Suppose that x is a periodic point such that $f^i(x) \notin (r,b]$ for any $i \geq 0$. Then $x \in [0,r] \cup (b,1)$. So $f(x) \in [f(0),b] \cup (f(0),f(b))$. But since $[f(0),b] \cup (f(0),f(b))$ is contained in the interval $[0,b]$, and $f(x)$ is not an element of $(r,b]$ we must have $f(x) \in [0,r]$. Similarly for any positive integer, k , $f^k(x)$ is an element of $[0,r]$. This implies that $f(x) = x$ since f restricted to $[0,c]$ is monotone increasing.

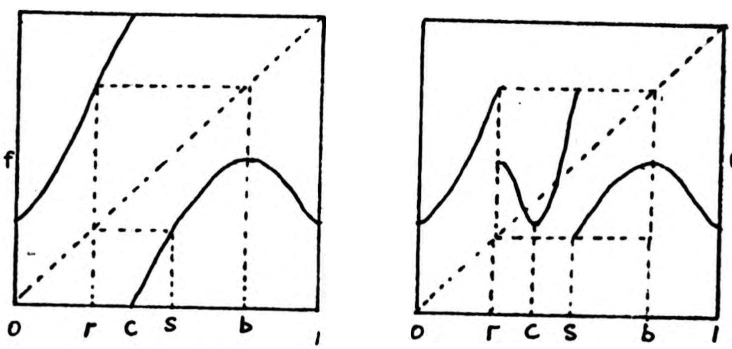
Choose $z \in B_f$. Then we must have z is greater than x because $f(z) > f(x) = x$. So $0 < x < z$ which means that $f(0) < f(x) = x$, but this gives a contradiction because $f(0) \geq z$.

It is easily seen that $f((r,s]) \cap (r,s] = \emptyset$. We will define $f_1 : [0,1) \to [0,1)$ by

$$f_1(x) = \begin{cases} f(x) & \text{if } x \notin (r,s] \\ f^2(x) & \text{if } x \in (r,s] \end{cases} .$$

Example.

Graph of f



Graph of f_1

Consider the map $\phi:(r,s] \to [r,s)$ defined by $x \to s-x$. It is easy to check that the map $\phi^{-1} \circ f \circ \phi$ restricted to $[r,s)$ satisfies case 2 of Proposition 5.2.2 or satisfies the conditions for Proposition 5.2.7. We can then see that applying Lemma 5.1.5 completes the proof.

5.3.6 Proof of the Proposition when f satisfies the conditions for case 3.

We know that B_f is non-empty. Given any element, z, of B_f we will show that we only need to consider the periods of periodic points in $[z,f(z)]$.

LEMMA. *If* x *is a periodic point then there exists a positive integer*, i, *with* $f^i(x) \in [z,f(z)]$.

Proof. Suppose that x is a periodic point and that for all i, $f^i(x)$ never lies in $[z,f(z)]$. Then x is an element of $[0,z) \cup [f(z),b] \cup [b,1)$.

If x is an element of $[0,z)$ then $f(x)$ is an element of $[f(0),f(z))$. Since $[f(0),f(z))$ is contained in $[0,f(z))$ and $f(x)$ is not an element of $[z,f(z))$, we have $f(x) \in [0,z)$. Similarly, $f^i(x)$ is an element of $[0,z)$ for any $i \geq 0$, which implies that $f(x) = x$. As in the proof of the Lemma in 5.3.5 we obtain $f(0) < f(x) = x < z$, which gives a contradiction to the definition of case 3.

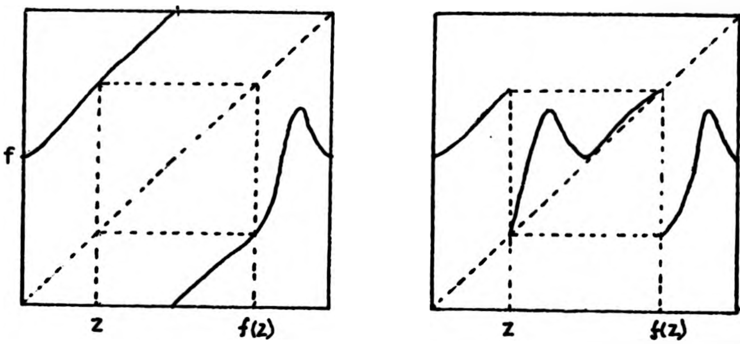
If x is an element of $[f(z),b]$ then $f(x)$ is an element of $[z,f(b)]$. Since $[z,f(b)]$ is contained in $[z,b]$ and $f(x)$ is not an element of $[z,f(z)]$, we have $f(x) \in [f(z),b]$. Similarly $f^i(x)$ is an element of $[f(z),b]$ for any positive integer i, which implies that $f(x) = x$, but this contradicts property 1 of Proposition 5.3.1.

If x is an element of $[b,1)$ then $f(x)$ is an element of $(f(0),f(b)]$ which implies that $f(x)$ is an element of $[0,z] \cup [f(z),b]$ and we obtain a contradiction by the argument above.

Now we must calculate the periods of the periodic points contained in $[z,f(z)]$. We know that z and $f(z)$ have period 2. It is easy to see that $f(z,f(z)) \cap (z,f(z))$ is empty, so we will define $f_1:[0,1) \to [0,1)$ by

$$f_1(x) = \begin{cases} f(x) & \text{if } x \notin [z,f(z)] \\ f^2(x) & \text{if } x \in [z,f(z)] \end{cases}$$

Graph of f



Graph of f_1

Since $z \leq f(0)$ and $f(z) \geq f(b)$, f_1 must map $[z,f(z)]$ onto itself. f_1 restricted to $[z,f(z)]$ is easily seen to be continuous and so we can apply Sarkovskii's theorem. Lemma 5.1.5 then tells us that for f we have 2×Sarkovskii's order inside $[z,f(z)]$.

So if we take $V = [z,f(z)]$ and U to be any neighbourhood not intersecting V we are finished.

5.4 PROOF OF THEOREM 5.0.2.

5.4.0 In this section we will complete the proof of the theorem. We only need to consider the following two cases:

1) $b > f(b) > f(0) \geq c$ and 2) $c \geq f(b) > f(0) \geq 0$. The other possibilities for f having been dealt with by Propositions 5.2.2 and 5.3.1.

5.4.1 LEMMA. Let f have the property that $b > f(b) > f(0) \geq c$. Then we have the following:

(i) If x is periodic then there exists a positive integer, i , with $f^i(x) \in [c,1)$.

(ii) There exists a unique point, $r \in [c,b]$, with $f(r) = c$.

(iii) $f([c,r)) \cap [c,r) = \emptyset$.

Proof. Let x belong to $[0,c)$. Then $f(x) > f(0)$ and as $f(0) \geq c$ we must have $f(x) \in [c,1)$. So statement (i) is clearly true.

Since $f|_{[c,b]}$ is continuous and $f(c) = 0$ and $f(b) > c$ we can obviously find a point r in $[c,b]$ with $f(r) = c$. It is unique because $f|_{[c,b]}$ is strictly monotone. Thus statement (ii) is true. Moreover, $f([c,r)) = [f(c),f(r)) = [0,c)$ so $f([c,r)) \cap [c,r) = \emptyset$.

5.4.2 DEFINITION. Let f have the property that $b > f(b) > f(0) \geq c$. Then we define $g_1:[c,1) \to [c,1)$ by

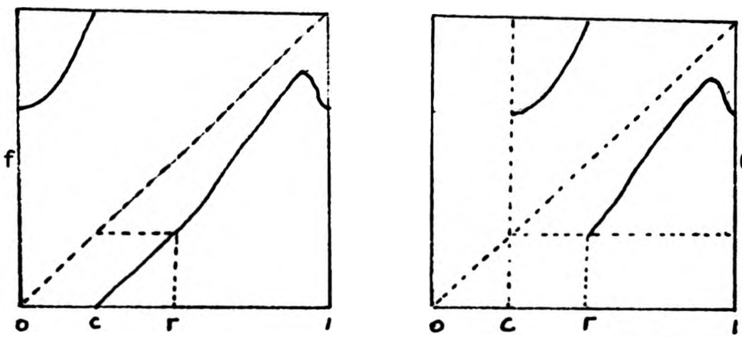
$$g_1(x) = \begin{cases} f(x) & \text{if } x \in [r,1) \\ f^2(x) & \text{if } x \in [c,r) . \end{cases}$$

5.4.3 NOTES. Firstly, g_1 is well-defined, since $g_1([c,r)) = [f(0),1)$ which is contained in $[c,1)$ and $g_1([r,1)) = f([r,1)) = f([r,b]) = [c,f(b)]$ which is contained in $[c,1)$.

Secondly, g_1 can be regarded as an element of $\mathcal{A}$

5.4.4 EXAMPLE.

Graph of f



Graph of g_1

5.4.5 LEMMA. Let f have the property that $c \geq f(b) > f(0) \geq 0$. Then we have the following:

i) If x is a periodic point then there exists a positive integer, i , such that $f^i(x)$ is an element of $[0,c)$.

ii) There exists a unique point, r , in $[0,c)$ with $f(r) = c$.

iii) $f([r,c)) \cap ([r,c)) = \phi$.

Proof. If x is an element of $[c,1)$ then $f(x) \leq f(b)$. As $f(b) \leq c$ we either have $f(x) \in [0,c)$ or $f(x) = c$. If $f(x) = c$ then $f^2(x) = 0$ which is an element of $[0,c)$. So the first statement is clearly true.

The proof of the second statement follows from the facts that $f(0) < c$ and that f is monotone on $[0,c)$.

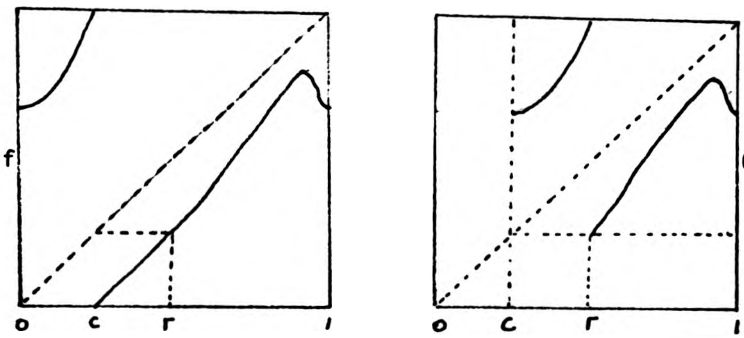
Since $f([r,c)) = [c,1)$ and $[r,c) \cap [c,1) = \phi$ the third statement is true.

5.4.6 DEFINITION. Let f have the property that $c \geq f(b) > f(0) \geq 0$. Then we define $g_1 : [0,c) \to [0,c)$ by

$$g_1(x) = \begin{cases} f(x) & \text{if } x \in [0,r) , \\ f^2(x) & \text{if } x \in [r,c) . \end{cases}$$

5.4.4 EXAMPLE.

Graph of f



Graph of g_1

5.4.5 LEMMA. *Let* f *have the property that* $c \geq f(b) > f(0) \geq 0$. *Then we have the following*:

i) *If* x *is a periodic point then there exists a positive integer*, i , *such that* $f^i(x)$ *is an element of* $[0,c)$.

ii) *There exists a unique point*, r , *in* $[0,c)$ *with* $f(r) = c$.

iii) $f([r,c)) \cap ([r,c)) = \phi$.

Proof. If x is an element of $[c,1)$ then $f(x) \leq f(b)$. As $f(b) \leq c$ we either have $f(x) \in [0,c)$ or $f(x) = c$. If $f(x) = c$ then $f^2(x) = 0$ which is an element of $[0,c)$. So the first statement is clearly true.

The proof of the second statement follows from the facts that $f(0) < c$ and that f is monotone on $[0,c)$.

Since $f([r,c)) = [c,1)$ and $[r,c) \cap [c,1) = \phi$ the third statement is true.

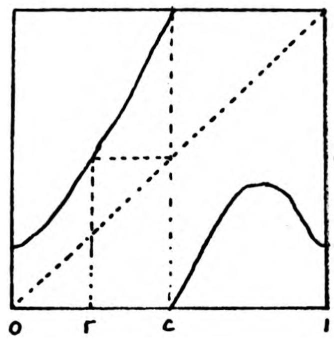
5.4.6 DEFINITION. Let f have the property that $c \geq f(b) > f(0) \geq 0$. Then we define $g_1 : [0,c) \to [0,c)$ by

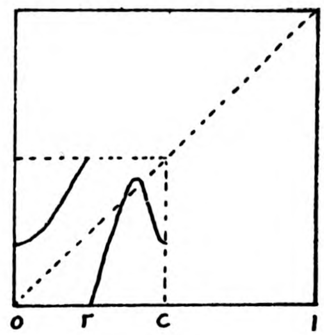
$$g_1(x) = \begin{cases} f(x) & \text{if } x \in [0,r) , \\ f^2(x) & \text{if } x \in [r,c) . \end{cases}$$

5.4.7 NOTES. 1) g_1 is well-defined.

2) g_1 can be regarded as an element of $\mathcal{A}$.

5.4.8 EXAMPLE.





5.4.9 PROOF OF THEOREM 5.0.2.

We have shown how to construct g_1 when $b > f(b) > f(0) \geq c$ or $c \geq f(b) > f(0) \geq 0$. Lemmas 5.4.1 and 5.4.5 combined with Lemma 5.1.5 tell us that in order to calculate the periods of periodic points of f we only need to consider the periodic points of g_1 .

If g_1 satisfies the conditions of Propositions 5.2.2 or 5.3.1 we are finished.

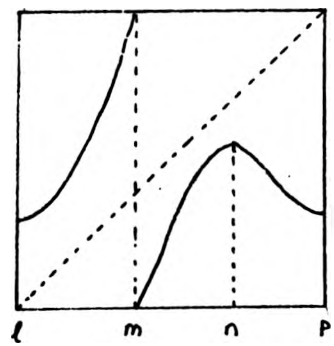
If g_1 does not satisfy these two Propositions we can construct g_2 in the obvious way.

To complete the proof we only need to show the following.

LEMMA. Let f have a periodic point x of period n . Suppose that g_i does not satisfy the conditions of Propositions 5.2.2 or 5.3.1 for all positive integers.

<u>Then if</u> y <u>is a periodic point of</u> f <u>it has period</u> n .

<u>Proof</u>. We will consider g_k , where k is an integer greater than $2n$.



Graph of g_k .

It is clear that $g_k\big|_{[\ell,m)} = f^i\big|_{[\ell,m)}$ for some i.

Similarly $g_k\big|_{[m,p)} = f^j\big|_{[m,p)}$ for some j.

So Lemma 5.1.5 tells us that every periodic orbit of f has period $aj + bi$, where a and b are non-negative integers. Since $k > 2n$, we must have $i+j > 2n$. So either $i = n$ or $j = n$, because g_k must have a fixed point corresponding to x . However $g_k\big|_{[m,p)}$ cannot have a fixed point because then g_k would satisfy Proposition 5.2.2, so we must have $i = n$.

Now suppose that y is a periodic point of period m . We can choose k greater than twice the maximum of n and m , and by the above argument obtain that $i = n = m$.

APPENDIX

In this appendix we give Bowen's definition of topological entropy.

Let (X,d) be a compact metric space. Let $f:X \to X$ be a continuous map.

DEFINITION. A set $E \subset X$ is called (n,ϵ)-separated for f if whenever x and y belong to E we can find an integer, k, with $0 \leq k < n$ such that $d(f^k(x), f^k(y)) > \epsilon$.

DEFINITION. Let $K \subseteq X$ be closed. Let $s_n(\epsilon,K,f)$ denote the largest cardinality of any (n,ϵ)-separated set E contained in K.

DEFINITION. Let $s_f(\epsilon,K) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\epsilon,K,f)$. Let $h(f,K)$ equal $\lim s_f(\epsilon,K)$. Then the topological entropy, $h(f)$, of f is defined to be $h(f,X)$.

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